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ENS Lyon is not going to give any approbation or disapprobation about the thoughts expressed in this dissertation. They are only the author's ones and need to be considered such as.

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Historical Events

Three funds from bank BNP Paribas freezed on August the 9th, 2007. These funds were made of United States mortgage loans, the so-called subprimes, structured in asset-backed securities (ABS), such as collateralized debt obligations (CDO). The French bank was unable to price the assets of the funds, because Mortgage securities' transactions were suspended. The funds came back to the market at the end of the same month, with reasonable losses, around 1%. At the same time, the insurance company AXA was facing similar difficulties. On august the 9th, the French index CAC40 lost 2.17% though there was no other annoucement in France. The commodity market was also deeply impacted. Why market's reaction was so large?

The freezing of BNP Paribas was a warning signal for international markets. Markets were less sensitive to the problems of U.S. investment banks like Bear Stearns. Indeed, a few days before, the shares' prices of two funds from Bear Stearns droped to zero. But it was known that Bear Stearns had invested in the U.S. laverage housing. These speculative investment funds were based on the hypothesis that U.S. housing prices would climb, and that would help the banks recover risky loans. The reputation of BNP Paribas was affected by the freezing. This also suggested that any of the international banks, funds or inurance companies could have invested in the subprime market. For example, a rush to withdraw cash from Northern Rock, the British mortgage lender, started on September the 17th, when Bank of England proposed to provide emergency financing to this bank if needed. The banks themselves were fearing each other, as shown by the interbank rate Euribor, which started to raise from August 2007 and reached in December 2007, 95 basis points more that the European Central Bank's refinancing rate, instead of the usual 15-20 points, illustrated on Figure 1. Obviously, such a high interbank rate highly affected banks, because they had to borrow at higher rates than the rates of the long term loans they had provided. This was also called the "liquidity crisis".

By the end of september 2007, large business banks had suffered substantial losses mainly due to the depreciations of assets. Among them, the investment bank

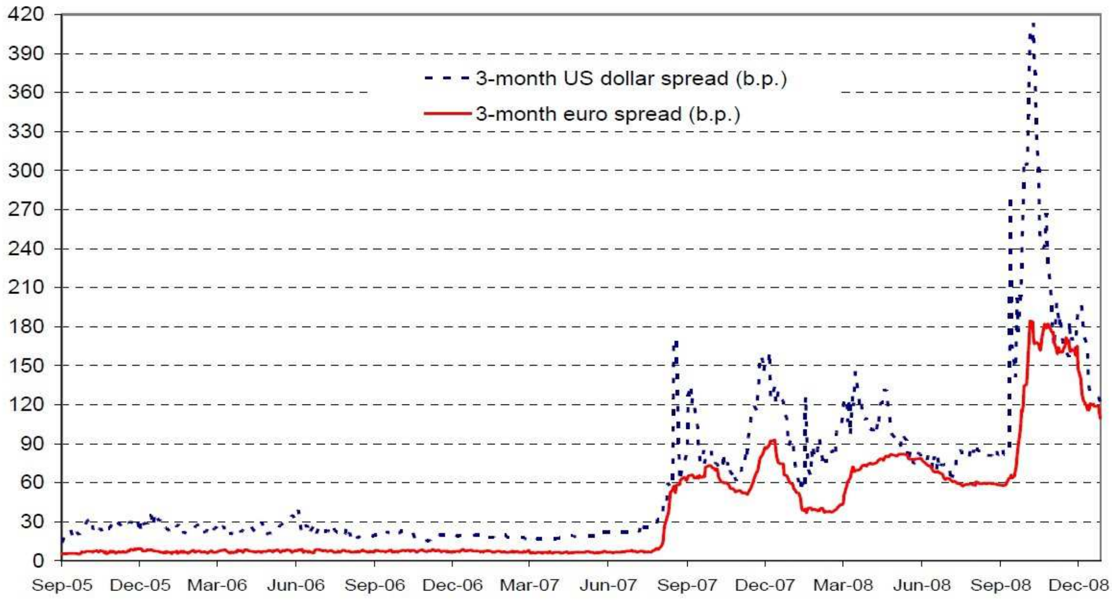

Figure 1: Interbank rate spreads (basis points, daily data) [a]

[a]Source: Angelini, Nobili, and Picillo (2011)

Merrill Lynch announced the highest loss of 8.4 billions USD[1]. However subprime mortgage originations had reached their maximal value in 2006, to represent 600 billions USD[2] in 2006. The comparison of losses between major investment banks in 2007 and the amount of the subprime market helps to understand that there was still a tremendous amount of subprime products in the whole economy, even if more concentrated in the U.S.. From the beginning of 2007 to April 2008, the major international losses or writedowns were, in billions USD[3]: Citigroup ≤ 40, UBS and Merrill Lynch ≤ 36, AIG ≤ 20, Bank of America and IKB Deutsche ≤ 14, Morgan Stanley and HSBC ≤ 12, JP Morgan Chase, Washington Mutual, Barclays, Ambac, Deutsche Bank ≤ 9.

[1]source: Bloomberg

[2]source: Inside Mortgage Market

[3]sources: Bloomberg and Reuters – Estimations of losses differ according to information sources.

Massive losses kept increasing due to the deepening of the subprime mortgage crisis. In March 2008, Bear Stearns was unable to find any refunding: the liquidity of the bank falled from 18 billion USD on March the 10th, to 2 billion USD on March the 13th. Finally, the Federal Reserve did not accept to provide a loan directly to Bear Stearns and organized a merger agreement with JP Morgan Chase in a stock swap, worth only 2 USD per share. Finally, the price per share reached 10 USD thanks to a class action of the shareholders. Even this low price of 10 USD per share remains negligible, compared to the value per share in January 2007: 172 USD, or in February 2008: 93 USD. The 29 billions USD loan made by the FED to JP Morgan to buy the assets of Bear Stearns was a non-revolving term loan maturing at the end of the month. Because Bear Stearns assets were illiquid, the low price per share was designed to guarantee that JP Morgan could refund the FED. That's why the chairman of the Securities and Exchange Commission, Christopher Cox, explained that a lack of confidence rather than a lack of capital was mainly responsible for the collapse of Bear Stearns: "capital is not synonymous with liquidity". Indeed, the SEC was subject to many attacks about its role in the collapse of Bear Stearns, and Christopher Cox answered that the investment bank was enough capitalized even when JP Morgan bought it. Among the large shareholders of Bear Stearns, in 2007, there were rich individual investors, as well as a number of investments funds and banks, among which Morgan Stanley, Barclays, UBS. We should pradoxically recall that in 2007, Bear Stearns was the "Most Admired" securities firm in Fortune's "America's Most Admired Companies" survey. Its ratio capital-assets at the end of the year 2006 was larger than 18%.

Just one month later, a major investment bank collapsed. Rather than a lack of liquidity, the decrease of assets' prices provoked the collapse. Lehman Brothers' stocks were deprecetiated by 73% of their value between January and July 2008. The laverage ratio was lower than the one of Bear Stearns, around 4% in 2003 and it decreased to 3% in 2007. On September the 9th, 2008 the price of the share of Lehman Brothers falled by 45%, because negociations about an hypothetical Korean buyer had been cut off. The same day, the S&P500 went down by 3.4%. On September the 15th, Lehman filed for Chapter 11 bankruptcy protection, and the Dow Jones lost also 4.4%. On September the 22th, 2008 negociations led to a proposal of Barclays, who was supposed to buy the main activities of Lehman Brothers, including the nice building in Manhattan, which indeed constituted the main part of the transaction (960 millions USD) with the other buildings in New Jersey. Again assets were worth nothing. Furthermore, even the real estate was significantly undervalued. Because of this bankruptcy, Lehman's mortgage securities would be liquidated: 4.3 billions USD. This provoked a rush on selling Commercial Mortgage Backed Securities. They were also two large falls in the Dow Jones, -6.9% on September the 29th, and -7.3% on October the 9th, the seventh consecutive day of decrease. A large number of banks, funds, all over the world (Europe, U.S. and Asia) had invested in or were secured by Lehman's assets, and their shares were devaluated in response to their exposures, which generated large losses for all of them. This was the end of the "too big to fail".

The origins of the subprime story

Real estate market

The development of the suprime market was the mix between a quite old state policy in the United States (Clinton, Bush), which aimed at helping the inhabitants to become owners, the regulation, and the behavior of mortgage U.S. specialists, (Fannie Mae and Freddie Mac - FHA was also insurer), and the "shadow banking system". The arrival of a large number of financial intermediaries on the mortgage market decreased the costs of lending, as studied by Dell'Ariccia, Igan, and Laeven (2012). After the dot-com bubble, growth had picked up again in the U.S. thanks to the boom of the real estate market, the consumption of the households and the low refinancing rate of the FED post September-11th. Indeeed, there was a very low unemployment rate in the U.S. from 2002 to 2007. Schwartz (2008) explains this low unemployment, at least partially, by a high employment in the building sector. There was a high demand on the market for housing because banks had to make houses affordable (especially through low interest loans). There were 3 types of mortgage loans depending on the quality of the borrowers: "prime" correponds to the best quality of borrowers, “Alt-A” intermediary, and “subprime” the worst. The quality of a borrower was based on many criteria, such that, the income, the credit history, the health coverage, the familial status, etc... The subprime mortgage originations by year expanded from 94 billions USD in 2001 to 664 billions USD in 2005 and 600 billions USD in 2006. Among the main subprime mortgage originators[4], there were HSBC, New Century Financial, Countrywide, Citigroup, which were also affected by large losses. Shiller (2008) analyzes booms in real estate prices overall the U.S. until 2007. Prices had never decreased from 1957 to 2007, even if there were disparities between the different states. As a consequence, no-one was expecting that foreclosures could not cover the total amount of loans. The share of subprime loans over the total rised from 2% in 2002 to more than 13% in 2005, thanks to a low director rate from the Federal Reserve until the end of year 2004, around 1%. The banks or financial intermediaries proposing loans could keep considerable laverage on the rates of those subprime loans while risky households were still able to pay. To crown it all, this created a boom in prices of the real estate, illustrated on Figure 3. Mortgage loans were guaranteed by real estate prices, as a consequence banks could not expect to make losses even if households would default.

The Federal Funds rate started to rise in 2005 to reach 5% in 2007. Obviously rates of variable rate loans (or, even worse laveraged loans) highly increased. The increase of the interest rate was likely to produce a delayed effect on borrowers with variable rate loans, because usually first the duration of the loan increases and then the monthly payment. However, risky borrowers started to default even before the repercussions of the increase of the Federal Funds rate on subprimes' interest rates. These defaults are probably due to the low quality of borrowers as proposed by Sorbe (2009). Actually, the boom in housing prices lead to a global decrease in denial rates, according to Dell'Ariccia et al. (2012). The delayed effect of the increase of the Federal Funds rate can be also explained by some original loans, for example the “interests only”, when the borrower only pays the interests on the loan for the first $2-3$ years of the loan; “balloon payments”, where the whole amount

[4]source: Inside Mortgage Finance

is repaid at the end, and also “neg-am”, when the monthly payment is lower than the corresponding interests, which makes the amount of the loan increasing with time. For a detailed analyzis of these original loans in the subprime market by state, see Chomsisengphet, Murphy, and Pennington-Cross (2008). As far as the default rate was low enough, profits coming from these loans were high enough to keep considering these subprime mortgage loans as a very profitable market. Moreover, investors were looking for financial products with high yields, and they were rare, because the FED was keeping a low refinancing rate. First financial insights on the subprime problems happened at the first quarter of year 2007 when defaults started to increase significantly. Until 2006, the average rate of default on all mortgage loans was around 5%, and the average rate of foreclosure on the same loans was about 1%. Both rates started to rise in 2006 to reach significant higher values in 2008: 6.4% of defaults and 2.5% of foreclosures. The rate of foreclosures kept increasing in the following years, as illustrated by Figure 2. Banks decided to foreclose and the demand went down: the bubble of housing prices collapsed in 2006, Figure 3.

There are to sensitive remarks with foreclosures.

- Due to the collapse of housing prices, some households were in “negative equity”: when the price of the house is lower than the remaining part of the debt. Within the U.S., the states’ regulations are different about foreclosures. Some states highly protect the borrowers, and some others protect the bankers: when there is foreclosure, and when the value of the house is lower than the remaining part of the debt, some states protect the borrowers from prosecutions even if the loan is not fully repaid by the price of the house (Harding, Miceli, and Sirmans, 2000). This kind of laws obvisouly encourage to default, as explained by Crouhy, Jarrow, and Turnbull (2008). Among these protecting states, with the highest rates of foreclosures over the crisis, we can find: Arizona, Nevada, California, Florida.

- When borrowers were defaulting, and when home prices fell, it would have been more interesting for banks to not foreclose, but wait until the borrower pay something. Indeed, foreclose procedure are costly and usually when

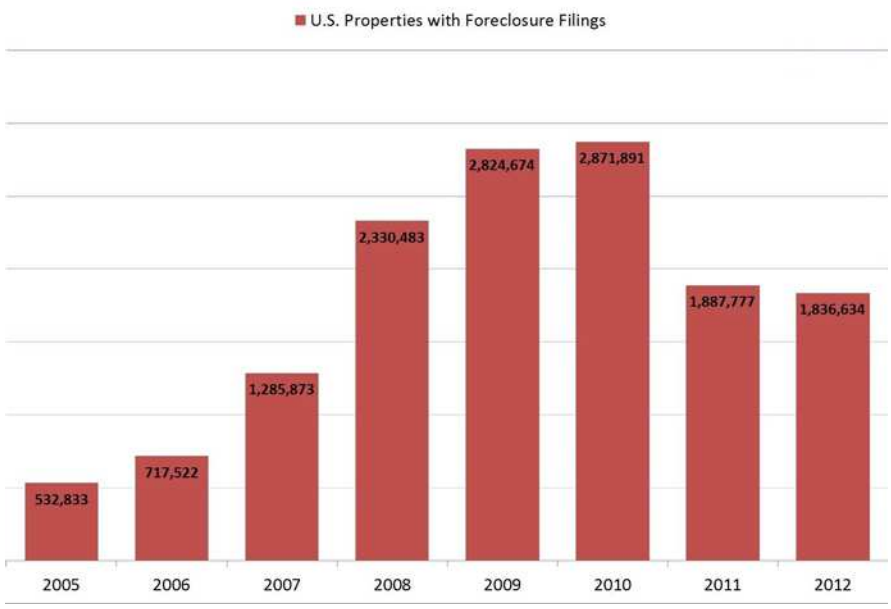

Figure 2: U.S. annual forclosure activity[a]

[a]http://www.realtytrac.com/content/foreclosure-market-report/2012-year-end-foreclosure-market-report-7547

selling a house this way, the bank gets between 40% and 60% of its value, as shown by Blundell-Wignall (2008). Foreclosures were very interesting when housing prices were still climbing, but became less profitable when prices decreased. Piskorski, Seru, and Vig (2010) explain that secritization is partially responsible for this high level of foreclosure.

Securitization

Securitization is considered as the main vector of contamination of the crisis. Securitization aimed at transforming illiquid assets, such as mortgage loans, car loans, students loans, even credit cards loans in a variety of marketable products with different yields and risks, possibly with insurances. Those products were rated by the well known agencies (Moody's, Standard&Poor's, Fitch), and were offered to all types of investors. Subprime securitized assets were appreciated because of their high yields. In the U.S. the issuance of securitized assets increased

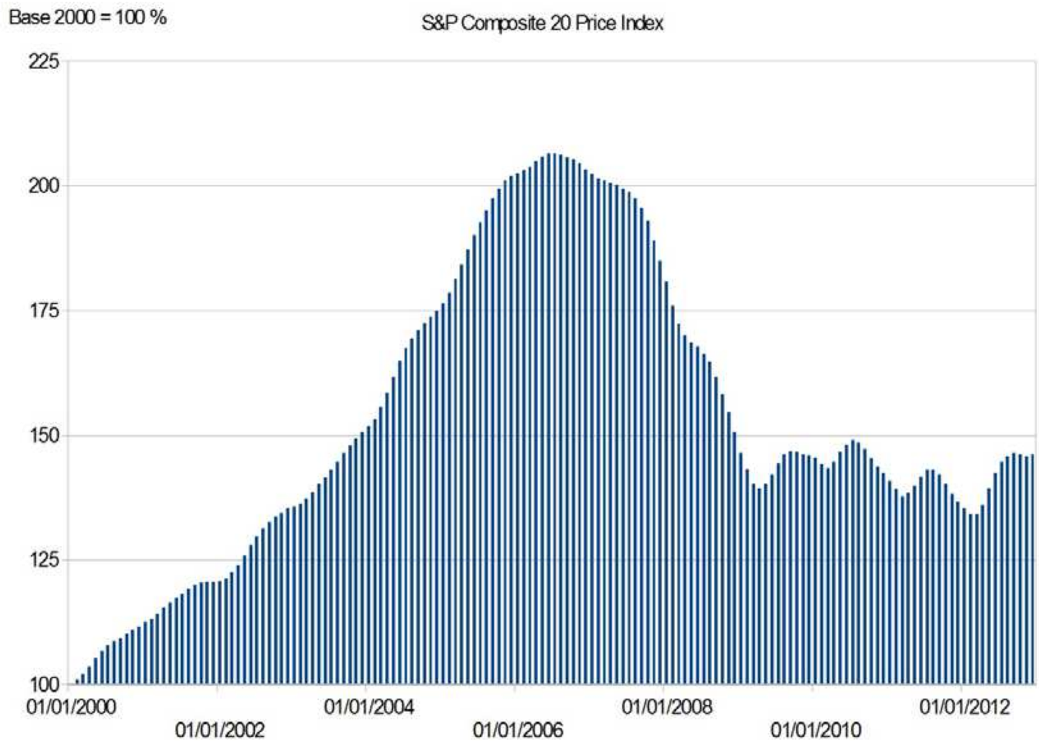

Figure 3: U.S. Case-Shiller home price index[a]

[a]http://www.standardandpoors.com/indices/sp-case-shiller-home-price-indices/en/us/?indexId=spusa-cashpidff–p-us—-

from 126 billions USD in 1985 to 2.7 trillions USD in 2005, mainly composed of mortgage backed securities[5]. Concerning mortgage loans, (75% of prime loans) 46% of subprime loans were securitized in 2001, and in 2006, 75% of subprime loans (86% of prime[6]). Added to the growth of the suprime loans, this shows that a high number of risky products appeared. It is considered that those products were sophisticated enough to hide their "risky side". Moreover, with the hypothesis of increasing housing prices, mortgage backed assets could be considered as very safe assets. Securitized assets (called Residential Morgagte Backed Securities RMBS and Commercial Mortgage Backed Securities CMBS) made of mortgage subprime loans were largely profitable until 2006, which encouraged investors from all horizons to

[5]Sources: Ginnie Mae, Freddie Mac, Fannie Mae, Inside MBS&ABS

[6]Source: Inside Mortgage Finance

buy them: big banks, big insurance companies, investment banks, private investors. Until 2007, MBS and CDO were AAA rated by the agencies. Moody's started to lower the credit ratings on subprime products at the end of the year 2006. In the following, all rating agencies lowered the ratings of a large number of assets. For example from 2005 to 2007, Standard&Poor's downrated 66% of CDO made of ABS, among which 44% to the speculative grade. The same way, 17% of the RMBS were downgraded, 8% to speculative grade[7]. One problem of these assets was the short maturity. Assets-backed securities could be short term, "commercial papers" ABCP, or long term: obligations. Initially, short loans (less than 3 months) were transformed in ABCP, while long term loans became obligations. From 2006, some housing loans (RMBS CMBS) became commercial papers. The issuance of ABCP highy increased, as shown on Figure 4.

At the end of maturity of each short term asset, the issuer must sell a new asset to pay back the previous one. This kind of asset (ABCP of mortgage loans) allowed to benefit from the difference between long term rates and short term rates. They were especially interesting when long term rates were subprime rates. These products were usually created by Structured Investment Vehicles (SIV), a kind of financial companies designed to earn the spread between assets and liabilities. Those companies were often created by banks but functionning under a lighter regulation than banks. The counterparty of such a practice is the risk of liquidity. When assets mature, if no one buy the new assets, prices collapses. Indeed when mortgage loans were downgraded by rating agencies, the corresponding ABCP were also downgraded. Since the main demand was for AAA products, SIV became unable to sell their assets. As a result, banks had to provide liquidity to their SIV, that's what happened for HSBC, Citigroup, Société Générale, etc... This also explains the sharp decrease in the issuance of ABCP post August 2007, see Figure 4.

Maturity risk can highly accelerate price drop: a financial intermediary looking for liquidity can easily collapse, like Bear Stearns. When markets became aware of the high risk of the products, banks were not willing to lend to others, in order to

[7]source: Standard and Poor's (2008)

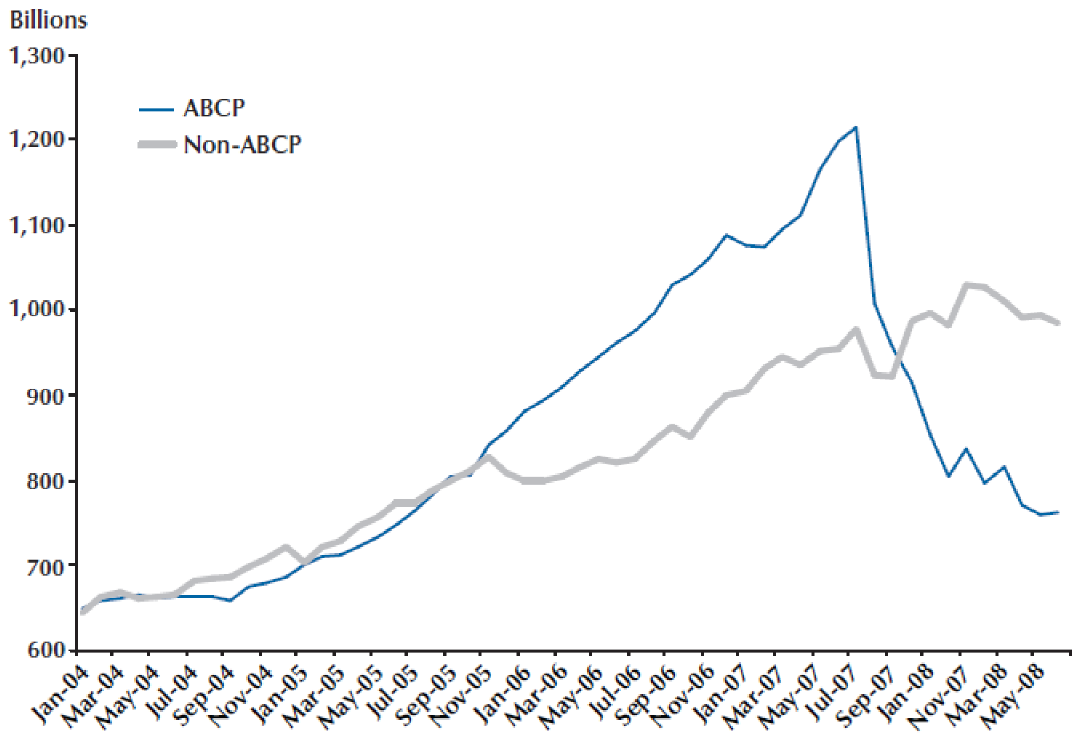

Figure 4: Issuance of commercial paper (ABCP) and long maturity ABS (non-ABCP)[a]

[a]Source: Mizen (2008)

avoid injecting cash in junk assets, and never be refunded. That's how subprime CDO helped the interbank market to dry up.

The securitized assets made markets very sensitive to any information. To conclude about securitization, it is very impressive to see how fast the financial crisis developed (mid 2007 to mid 2008) while mortgage loans defaults and foreclosures were still at the beginning ot their endless rises, as illustrated on Figure 2.

Rating the subprime

During the crisis, rating agencies lowevered the grade of a large number of products, and sometimes AAA-assets became junk bonds in a few weeks. This huge difference of rating is unsual, and cannot be the only effect of the macroeconomic context. This proves that risk was largely underestimated. Because of securitization, mortgage backed assets were complicated enough to become opaque for investors: it was difficult to figure out what kind of borrowers and houses were really concerned by the assets. In addition, what is the risk provided by a number of houses and borrowers all over the U.S.? Because of this lack of information, the only judgment of investors was the rating of the assets.

CDO had to be rated to be offered to investors. There were only 3 global rating agencies, and their rating mechanisms were well-known from the issuers. As a consequence, they could propose particular structures of CDO to obtain the AAA rating. Issuers needed to sell AAA-rated CDO, because there was a high demand for those products. Indeed money market funds were only intersted by AAA assets and mutual funds were also looking for high rated assets. As a consequence, issuers prepared their CDO to get good ratings. To rate the products, they had to pay the rating agencies. Therefore CDO with lower ratings would have not been interesting, because of the costs of production and costs of rating, and there would have been no demand. Besides the funds, banks were interested by AAA assets, because they were part of Basel II ratios. Banks were also interested by the securitization of lower graded assets, because it allowed them to remove a part of their assets from their balance sheets, and therefore satisfy more easily Basel capital requirements.

Rating Agencies were paid to rate products, and did not proceed to cross checks about origins of mortgages, which would have been more costly (Crouhy et al., 2008). The quality of subprime borrowers was decreasing from 2005, especially because of the increase of the Federal funds refinancing rate, but there was still a high demand for high yield products. This helped to maintain a high level of subprime assets production.

Too Big To Fail

Lehman Brothers collapsed in 2008. The Federal Reserve did not intervine to save the bank. This provoked a shock on financial markets, because no one expected it. The concept of TBTF (too big too fail) was broken. Nevertheless, later in the crisis, other major financial institutions were seriously damaged, but actually the Federal

Reserve Bank organized their rescue. For example, the Federal Reserve proposed a credit of 85 billions USD to the insurance company AIG to meet its collateral requirements, and then the U.S. Treasury prepared the bailout of the company, in September 2008, one week after Lehman's collapse. The Federal Reserve and the U.S. Treasury could not let any other big financial institution down without running the risk of a financial panic. Citigroup in November 2008 and Bank of America in January 2009 benefited from this TBTF policy.

This illustrate how large financial institutions received governments supports regarless of their management, owners and creditors, because the governments "recognize that the consequences for the broader economy of allowing a disorderly failure greatly outweight the costs of avoiding the failure in some way." That's how Bernanke (2010) explains the situation. According to Bernanke, TBTF institutions are subjected to "moral hazard", because creditors do not ask for enough compensations for risk. TBTF institutions may take more risk than they should. In addition, the TBTF concept penalize small institutions, because they are sujected to tougher market discipline than large ones. Kane (2000) had already explained that large financial institutions were "too big to discipline adequately". This is one of the unofficial incentives that led to banking megamergers.

These key issues of the subprime crisis strenghtened the determination of states to control what was really happening in finance.

Bâle III and others

Because states and regulation commissions (especially the SEC) were unable to predict systemic risk (Lehman Brothers), they try now to identify and to watch carefully those big financial structures that are disseminating assets all over the world. There are also discussions about the role of "lender of last resort" played by central banks.

First of all, the crisis happened while the implementation of Basel II Agreements was not complete, but at the beginning. The decisions of Basel II were supposed to be applied at the end of 2008 in Europe and at the beginning of 2009 for major

U.S. banks (but never happened in the U.S.). Solvency II is supposed to care about insurance companies, and should be fully implemented in 2014. The relevance of such measures is hard to evaluate regarding the crisis. Basel II agreements have sharpened the capital requirements against new forms of risks according to their importance: they include the previous credit risk from Basel I, but also add the operational risk (intern, there are numerous examples of computers' problems that generated huge distortions on market prices) and the market risk (losses arriving from problems or movements in the market, for example the increase of rate Euribor mid-2008). In addition, banks are supposed to proove that they have made the right choices to measure accurately the different risks: the "Internal Capital Adequacy Assessment Process". Because this regulation must be applied at the international level, everybody has to be able to understand the accounting of the others: banks must provide to any interested shareholder a set of informations about their capitals, which aims at helping the others to get an accurate view of the banks. This is called the "Market Discipline".

A large number of measures (also called Basel III) have been undertaken by the Financial Stability Board and the G20 at the end of year 2010 to reinforce the measures of Basel II. First, the ratios of Basel II are strenghtened: there must be an increase in quality and quantity of the capital requirements. Second, to avoid liquidity problems and specific risk problems, two new ratios are introduced, the Liquidity Coverage Ratio, made of the supposed "quick resale price" of the different assets of the bank; and a Net Stable Funding Ratio, an amount that must be larger than the needs of the banks when facing a specific risk. Third, a leverage ratio has also been introduced to limit the total exposure depending on the capital. Finally, there will be also a new macroprudential tool, depending on the market supervisor, which increases the requirements of major banks when market conditions degrade, to prevent systemic risk. All these measures should be progressiely applied to the banks from 2013 to 2019.

At the same time, in the U.S., a new law proposed by Obama's administration has been enacted on July 21th, 2010: the "Dodd–Frank Wall Street Reform and Consumer Protection Act". Indeed this Act is still not finished, because it was supposed to be controlled and finished (200 rules, 70 studies) by a set of regulators 18 months after the acceptation of the law, but it is not ready. This law strenghtens the power of the Securities and Exchange Commission, and creates two regulators: the "Financial Stability Oversight Council" and the "Office of Financial Research", both under government control. The new law also extends the types of financial entities (hedge, insurers...) subject to different procedures, like liquidation, control, settlements... The law also creates a number of Clearing Houses to avoid liquidity problems, and finally strenghtens the protection of investors.

Across the ocean, there is a challenging process to standardize the legislation (on financial supervision, commissions) between the 27 countries, and the creation of number of new regulators to apply Basel III (details in Perrut (2012)).

Consequences and cures

All these agreements (or laws) have been acted with good will, and they tend to avoid the major situations that led to the recent crisis. We can already question about their effects and their possible usefulness.

In response to Basel III, there will be a change in rates of loans issued by the concerned banks, and in the overall growth of loans. According to Cosimano and Hakura (2011), the average annual lending rate of banks will be increased by 16 basis points, and therefore the loan growth will decrease by 1.3% in the long run. They also explain that the effect on banks depend on the ability to raise funds, which differs from one country to another (easier in the U.S. than in Europe). Angelini, Clerc, Curdia, Gambacorta, Gerali, Locarno, Motto, Röger, Van den Heuvel, and Vlček (2011) quantify the effects of the new regulation on the output: each percentage point of increase in the capital ratio of banks should decrease the output by 0.09 %. The liquidity ratio should also produce a similar effect. On the opposite, the new regulation should dampen volatility. Allen, Chan, Milne, and Thomas (2012) also explain that credit could rarefy, but miscoordination costs of implementing the reform rather than the ratios themselves would account for this phenomenon.

The study made by Al-Darwish, Hafeman, Impavido, Kemp, and O'Malley (2011) shows the possible distorsions between Basel III and Solvency II. It seems that bank capital requirements in Basel III are tougher than insurance companies capital requirements of Solvency II. This could create a change in activities. Banks will also remove exposure from their balance sheets to meet capital requirements, they are likely to use securitization, and this might lead to "overcontentration of exposures in less regulated areas of the financial system". The structure of capital of banks is likely to change because some assets are included in the capital requirements while some others are not.

Actually, larger banks could better integrate the costs of new regulations, and that could lead to a concentration of the activities. According to some representatives of banks, the consumers and the firms will absorb a part of the reform. For example, new saving products might be created with penalties for early withdraw, in order to obtain guarantees on the capital requirements.

At the end of 2012, the SEC postponed the implementation of the capital requirements ratios later than 2013. In February 2013, due to the poor economic situation in Europe, the liquidity ratio has been decreased and the implementation of the two ratios (NSFR and LCR) has also been postponed to 2015. Regulators were fearing that these ratios would create a credit crunch to firms. The SEC also abandoned the project to supervise monetary funds.

History repeats itself

The implementation of Basel II, Basel III and Solvency II has definitely put a stop to the autoregulation and the liberalization of the banking sector which truly started in the eighties, to end up a little before 2000. These laws (Basel III and especially Dodd-Franck Act) strongly recall the Sarbanes-Oxley Act of 2002. After Worldcom and Enron scandals, the U.S. had implemented a very strong law about listed firms on the U.S. stock exchanges. This law is designed to avoid corporate and accounting scandals. Therefore it is supposed to protect investors from operational risk. As in the Dodd-Franck Act and Basel III, similar measures have been implemented to improve transparency (accounting standards, regular auditing procedure) and establish emergency procedures and settlements for firms. Post crises, it seems that regulators favourably respond to public opinion by making laws that guarantee to never remake the crisis. Each crisis, there is a strong response to the entities (banks, firms, insurance and audit companies...) that have endangered the whole economy, or at least, impacted shareholders and investors. Each crisis presented also a notion about systemic risk, last time there was the liquidation of Arthur Anderson, and this time it was, amongst others, the collapses of AIG, Lehman Brothers, Dexia...

A few years after the implementation of SOX, in 2007, there were numerous criticisms about the effect it had on small and intermediary firms. Indeed, the constraints applied to firms seemed to be too strong (multi annual reporting obligations, IFRS standards) and too costly (audit fees). As a consequence, many reports (among which Paulson (2006) and Bloomberg and Schumer (2007)) were calling for relaxation of the conditions of SOX. They explained that NYSE was loosing attractivity, by looking at the number of Initial Public Offerings on other international stock exchanges, especially the Alternative Investment Market from London. Actually this idea of the decrease of the number of IPO was manipulated. There were a high number of IPO on small deregulated stock exchanges (AIM, Alternext), corresponding to very small and risky firms. Major firms kept introducing on the NYSE, except Gazprom introduced on the London Stock Exchange, which is easy to conceptualize. This question about relaxing SOX was shelved by the recent subprime crisis.

In their book, Reinhart and Rogoff (2009) tend to demonstrate that this situation of crisis has been experienced many times before, even if persons, locations, technologies and times change. They identify debt sovereign crises (internal and external debts), inflation episodes, exchange rate crises, and banking crises. They consider 66 countries over 8 centuries. They show that crises seem to be inevitable. Because crises are infrequent enough, the actors (investors, regulators) forget about the signs of "overheated" economy and think they will not reproduce their old mistakes, thinking that "this time is different". All crises are preceded by over accumulation of debts, from consumers, states or banks. Usually this accumulation of debt goes hand in hand with growth but ends in failure. Banking crises usually provoke a high increase in sovereign debts (80% in 3 years – which was an interesting prediction for numerous countries today). They underline that most developed economies are supposed to know about and therefore avoid the risk of sovereign debt crises. Consequently they hold the trust of investors, but this may be an illusion, they are vulnerable as emerging economies. Real prices of housing are a good index for banking crises. To conclude, international institutions could decrease risks by providing large sets of informations about the states (intern and extern debts) and on banks.

Design of the research

All along the dissertation, we try to model key features of the past crisis. A noticeable characteristic is the contagion of the subprime sector to the whole banking system. The two first chapters deal with this question, and present the network and contagion approaches of the financial system. Another distinguishing feature of the crisis was the high increase of the credit supply associated to a kind of bubble on housing prices. To analyze this bilateral phenomenon, the three last chapters deal with the influence of the investment and the collateral constraints on the – possibly – bubbly price of the assets. Chapter 1 (respectively chapter 3) reviews the fundamental literature that led to the research work of chapter 2 (respectively chapters 4 and 5).

Financial networks, chapters 1 and 2

The connectivity of financial activities is a concrete way to address the question of the systemic risk. When markets participants are linked together because of counterparties, contracts, securitization, the whole system is possibly affected by a single default. How conceptualize this question of contagion? Especially, what initiates the financial problems, how idiosyncratic shocks transmit to a complex financial system, depending on its structure? From a macroeconomic point of view, the regulator tries to control and limit the activities of financial entities. The prudential regulation policies have a local effect by influencing the balance sheets of the agents, the capital requirements, etc... To which extent those policies modify the global structure of the financial system, and how can we test their efficiency on the resilience of the system? To answer all these questions, we adopt a network approach of the financial system.

Different mechanisms have been analyzed in the literature on contagion, including for example, bank runs (Diamond and Dybvig, 1983), short-sales (Allen and Gale, 2000), risky assets (Dasgupta, 2004). Very close to real phenomenons, they are restricted to simple financial systems, compared to the complexity of the contemporary finance world. Since 2007, a new branch of the literature emerged, based on the network representation of the financial system. The novelty of this approach is to capture the intricacy of links between financial agents, the number of links, their financial weight, their maturity... For example, this new approach allows to analyze systemic risk (Gai and Kapadia, 2010), prudential regulation (Anand, Gai, and Marsilli, 2012) or herd behaviors (Cont and Bouchaud, 2000). This evolution of the literature is presented in the first chapter of the dissertation.

Chapter 2 details our network model of the financial system and its implications. We model the interbank market as a financial network populated by a large number of agents, which may represent banks, insurances companies, funds, financial intermediaries. These agents are supposed to be identically capitalized. We suppose that these agents make investments among others, because they expects profits on these investments. When they mature, the investments are supposed to be paid back by their destinators to their issuers. However, if the investments were not to be refunded, it would affect the balance sheet of the issuer, and maybe make him default. As a consequence, this issuer would not honor his own debts, and could propagate the default to his own creditors. This is our contagion mechanism in the network. The origins of defaults come from some agents who decide to strategically default, because their gains from defaulting exceed the amount of the fine. Given these informations, we can calculate the defaults in the network. We can also deduce the expected payoffs of the agents, and deduce there choices, in terms of strategic defaults and numbers of investments.

The regulator sets a prudential ratio, which limits the number of assets of the financial participants in terms of their capitalization. The resulting state of the network depends on:

- the time-horizon maximization of the agents,
- the capitalization level of market participants,
- the prudential ratio.

Depending on these parameters, the financial network may reach either a highly connected state (dense network), or a low connected state (sparse network). Intermediate situations, where the network is moderately connected are rare, because they tend to propagate defaults and reduce the payoffs of the agents, especially in short and medium time-horizon. Dense networks better withstand contagion, and increase the expected payoffs of agents, they also include a rare but systemic risk.

Limiting the assets of participants avoids very dense networks, and the net effects of this limit depend on the time horizon: it may reduce systemic risk for long time-horizon maximization of participants, but it may become counterproductive when agents adopt a myopic attitude. Finally, the prudential ratio is ineffective in short-run maximizations.

Collateral limits, investment opportunities, and bubbles

It seems that for the recent crisis, subprime assets were overevaluated. This may be linked to their high yields despite their also high risks. U.S. housing prices were also overestimated. For the previous crisis, many firms were overpriced, it was the dot-com bubble. For a long time, research has been tracking bubbles. Apart from the words, theoretical models do not generate bubbles except under particular assumptions, because the usual context (Santos and Woodford, 1997) does not help emerge bubbles on assets. Historically (Blanchard and Watson, 1982), the price of an asset represents the discounted sum of the incoming dividends. If the price of the asset is used as a collateral guarantee (Kiyotaki and Moore, 1997), it may deviate from its historical value. Especially, if the price allows to relax a debt constraint, many authors proved that the price of the asset is likely to include a bubble part (Kocherlakota, 2009). Chapter 3 traces the stakes of the theoretical literature on bubbles until very recent developments.

Chapter 4 extends the work of Miao and Wang (2011). They consider a production economy with a large number of firms, all having the same production function. These firms face stochastic investment opportunities. This means that they can not achieve their investments at each period, but they wait for the investment opportunity to happen. Because of this scarcity of investment, firms might be willing to invest more than their capital gains to reach the optimal level of investment, to wait until the next investment opportunity. This is made possible through borrowing. Nevertheless, the authors assume that borrowing needs guarantees. Because firms generate cash-flows at each period, it is possible to determine the prices of the firms, as the discounted sum of their incoming cash-flows (or dividends). To guarantee that the firms refund their loans, Miao and Wang (2011) consider that the firms pledge a small amount of their value to secure the loans. Precisely, a novelty of this model is the new collateral constraint: firms pledge a small fraction of their capital, and the debts are secured by the value of this small amount of capital. Their model shows that under specific collateral constraints and stochastic investments, the optimal solution of the market is to overprice firms to help them reach the optimal investment level. This pricing is interpretated by the authors as including a bubble.

We deepen this model by adding a net positive interest rate on the loans to the firms. Because it is usually assumed that increasing rates of the loans provokes the collapses or at least help tame bubbles, we want to examine wether a net positive interest rate on the loans influences firms' prices, and to which extent. This would account for a regulator's policy that could control bubbles through interest rates. We assume that this interest rate is fixed exogenously, and that the interests are paid to an external bank. In addition, we can determine an optimal interest rate if the bank optimizes profits. The results are not really convincing. The impact of the interest rates on the loans are negligible. This is mainly due to the very low debts amounts, and the firms borrow just when investing: debts can be interpretated as short-term maturity. In addition, we question about the existence of a bubble in the prices. It seems that the equilibrium pricing corresponds to an affine pricing, which does not include a bubble part. To overcome those obstacles, we decide to change the role of the debts.

Chapter 5 is the completion of the analyzis of the weaknesses of the previous model. Interest rates on debts do not play a crucial role, and debts amounts are too small. Because capital of firms is usually composed of equity and debts, we adapt this particular capital structure to the previous framework. Firms do not borrow when the investment opportunity happens, but have a permanent part of long-term debt in the capital. Firms face stochastic investment, and their debts are limited to a fraction of the prices of their equities. This is the equivalent of the mortgage loans, where loans were secured by the values of the houses. The loans are supplied perfectly elastically by a bank, which also optimizes profits by fixing the interest rate. In this case, we show that values of firms are very sensitive to interest rates variations. We show that the results are independant from the choice of the borrowing constraint: the results are identical with the constraint of the previous chapter (pledge a value of a part of the capital), and also with the most classical one (pledge a part of the value of the capital). Two equilibrium situations coexist: the standard case, with high equity, normal price and proportional debt, and the “binding” case, with lower equity, higher price and larger debt. In the second case, the equity has a higher yield, but the equilibrium prices are very sensitive to interest rates: increasing the interest rate highly affects prices and capitals of firms. For a particular relation between parameters, there exists a bubble in the firms’ prices, because the equity is negligible, but has a net positive price, which allows for borrowing, and therefore firms have a net positive capital.

This analyzis has been a great opportunity to study models dealing with financial crises. Using networks to capture propagation of defaults and systemic risk is very relevant, though it remains difficult to bring together all interesting contributions of different articles, because they often require conflicting assumptions. In that direction, enhancing models is possible, but this has to be done carefully, to preserve readability, tractability and their policy implications. An interesting variation of chapter 2 would be to introduce a small number of bigger agents, to catch concentration and TBTF situations.

Our bubbly contribution allowed us to understand how subtle it is to generate bubbles by a theoretical approach with infinitely lived agents. Instead of speaking about bubbles, it seems that we should focus of the dual role of prices: they represent but also generate value. In the one hand, models are able to catch overheating of the economy, on the other hand, up to now, crashes are not endogenous. In addition alternating bubbles and crashes constitutes a serious challenge to improve contemporary research.

Chapter 1

Contagion in financial networks: review of the literature

Contents

1.1 Networks topology

Contemporary finance is characterized by both high connections between financial entities and uncertainty concerning the connectivity of market participants in the financial system. A study of the IMF (2010) analyzes cross border funds between major financial countries. The size of flows does not rely on the size of the countries. Among the different countries with cross border funds, they distinguish net lenders, net borrowers and main conduits. For example, Luxembourg collects funds from Lichentstein and Cayman Islands (net lender) and distributes to large industrial countries: USA, UK, Spain, France, Germany... They enlight the complexity of transfers between banks and funds, depeding on the types of asset, their durations, ratings, etc...

Merton, Billio, Getmansky, Gray, Lo, and Pelizzon (2013) analyze the effect of sovereign and credit risk arising from different European countries on major international banks, brokerages and insurances economies. On Figure 1.1, they plot the financial network linked to Greece in August 2008. Greece was highly connected to numerous international banks, sovereigns and insurances. However, the authors also explain that connections were varying over time. They remarked that the U.S. were little exposed to European banks and sovereigns in March 2012.

Securitization was supposed to improve risk management according to Duffie and Garleanu (2001). It has been responsible for inconspicuously disseminating a common risk all over the financial network. As explained by Kane (2009), financial engineering helps financial institutions to get larger and more complex, which strengthen their political influence, and may also increase the volatility of portfolios that would not appear on monitoring technologies. In addition, trade volumes of markets participants highly increased over the last decade. This led to a worldwide connected financial network, conceptualized in Minoiu and Reyes (2013).

A detailed description of the federal funds market and the network of american banks, was made by Bech and Atalay (2010). Their study lasted from 1997 to 2007. They deliver an overview of the number of active banks each day, as well as the number and the amount of the contracts and the number of interbank links. This encourages to use a network structure to depict the financial system. A network

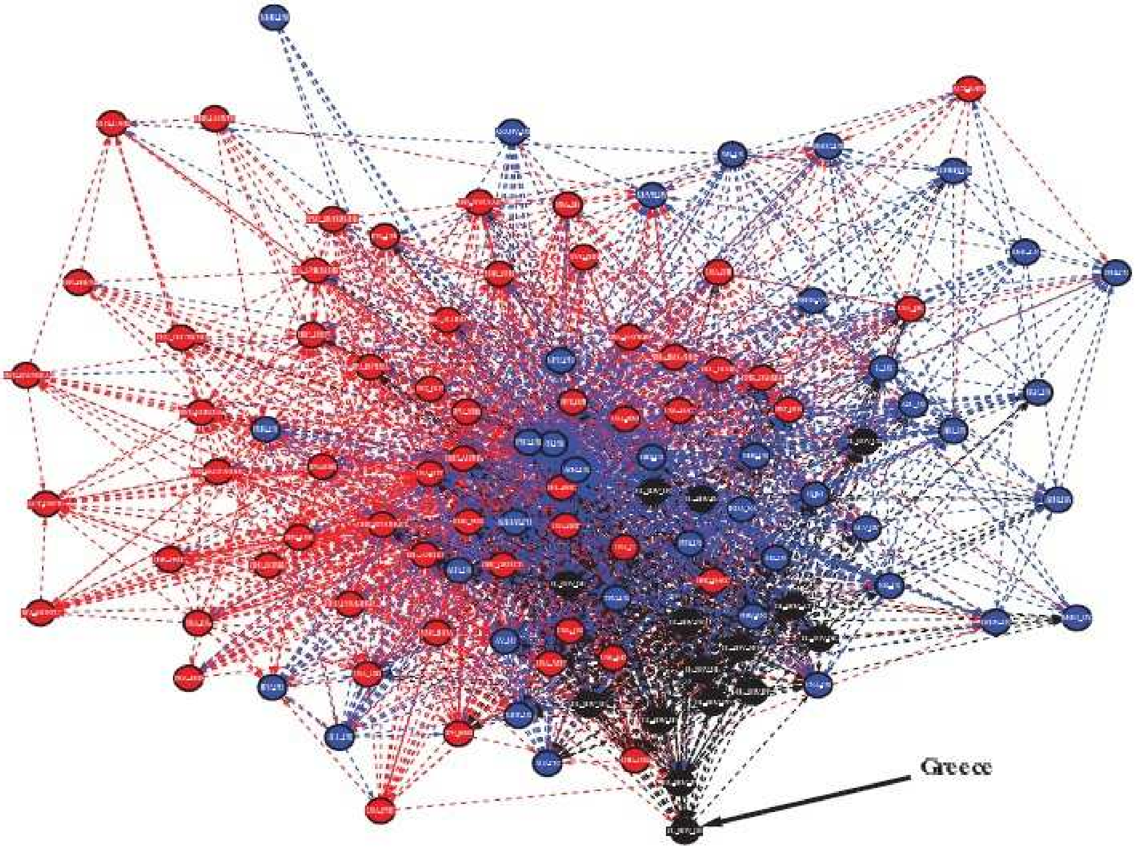

Figure 1.1: Connectedness of sovereigns (blue dots), banks (red dots), and insurance companies (black dots) to Greece, August 2008[a]

[a]Sources: Merton et al. (2013).

approach to model interbank links allows to represent the multiplicity of links between participants. It also captures the different paths or credit lines between banks, i.e. the sequences of different intermediaries that link any two participants together.

Allen and Babus (2008) present a literature review on the topic of networks, especially in finance. Though empirical literature justifies the use of networks to model financial systems, we mainly focus on the theoretical literature. Most of the authors on financial networks have studied the contamination of defaults depending on the structure of the network, and also how financial agents choose their links in the network. A large majority of articles concludes that financial networks with a more complete set of connections better withstand shocks than less dense networks. Logically, disconnected networks do not transmit failures. Allen and Babus also underline that authors have been more studying the effects of the network structure on the agents rather than the effects of the agents which constitute the network on the network itself.

Introducing a time dynamic model within a general network structure remains sensitive, unless there are tight constraints on parameters, such that the number of agents. When considering very small networks: the implication is rather micro orientated and the results are formulated by closed-form solutions. When studying larger networks, the properties of the networks are deduced from probabilities and the study of graph theory, as early developed by Erdös and Rényi (1960) and deepened by Bollobas (1985). We first present articles from the economic theory that introduce cross deposits, and model contagion through different mechanisms. Second, we show how the graph theory has successfully been applied to financial contagion.

1.2 Theory of contagion

1.2.1 A well-known approach

One of the seminal contributions in the literature is Allen and Gale (2000). They study how the banking system reacts to contagion in a two periods model. The economy is made of consumers, each of them owning one unit of consumption at time 0. A part of consumers will only consume at time 1 and the other part only at time 2. Consumers can store their endowment at the banks, which access two types of assets: "short" assets last over one period and have a net positive yield $r > 1$. "Long" assets last over the two periods and have a higher yield than short assets, $r_2 > r$. However, if sold at the end of the first period, they have a lower yield, which can be interpretated as a liquidation value, $r_1 < 1$. Consumers are identical at time 0 and there is a probability p that they only consume at time 1, and $(1-p)$ that they only consume at the end of the second period, $t = 2$. Consumers only know their types – early or late consumer – at the end of the first period, (otherwise the realization of p would be known at time 0). The global economy is composed of 4 regions, and each region has a bank (or a number of identical banks). The probability p can take a low value p_L and a high value p_H, which distinguishes when there might be a lot of early consumers p_H or a lot of late consumers p_L. There are two states of nature, such that two regions have either a lot of early consumers or a lot of late consumers, and the two other regions have the other type. The social planner wants to optimize the utility of the consumers, $u(c_1)$ in period 1 and $u(c_2)$ in period 2. Regions with a high number of early consumers face a high demand for liquidity at time 1 while regions with a high number of late consumers face a low demand for liquidity at time 1. Banks ignore the types of their consumers before period 1. To face the demand for liquidity at time 1, banks are supposed to hold the short asset, while they would obtain a higher yield for late consumers by holding the long asset. If they face a larger liquidity demand at time 1 than their holding of the short asset, they will loose some money by selling the long asset. This standard framework had been previously introduced by Diamond and Dybvig (1983) to deal with bank runs: when depositors expect the other depositors to withdraw their funds, the optimal strategy is to withdraw before the others do. One of the new features of the model of Allen and Gale (2000) is the "cross deposits": the optimal allocation between both assets can be obtained by allowing transfers between the different banks. In this case, banks with a higher demand for liquidity at time 1 withdraw their deposits in banks with low demand for liquidity at time 1, and reciprocally at time 2. This is possible whatever the number of counterparties: banks may hold deposits in all other banks, or just a part of them. The authors model the effect of a shock of liquidity. Suppose that there is an excess demand for liquidity in period 1 in one region: the bank of this region will withdraw its deposits from the other bank(s). In addition, to face the excess of demand for liquidity, the bank needs to liquidate some long assets. Because of this liquidation, it happens that assets of the bank are loosing some value: $r_1 < 1$. If the bank is still able to guarantee at least c_1 units of consumption to late consumers, the late consumers will not withdraw their holdings in the bank at time 1. The amount $(c_2 - c_1)(1 - p_H)$ can be interpretated as a capital buffer. On the opposite, if the

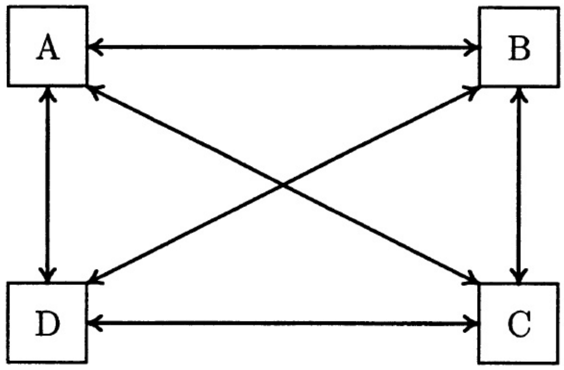

Figure 1.2: Complete network[a]

[a]Source: Allen and Gale (2000).

decrease in assets' values do not guarantee at the late consumers at least c_1 units of consumption, they decide to withdraw at time 1 and the bank must liquidate all its assets, the bank goes bankrupt. At time 1, the bank will also face the withdrawals from other banks. Since all assets are liquidated, the other banks will also loose some money because their withdrawals will also be devaluated. Because of this depreciation, the other banks might get some trouble: if the capital buffer of the other banks is lower than the loss on their deposits, they also go bankrupt. The response of the whole system depends on the type of the market structure: when banks hold deposits over all other banks, the network structure is said "complete", cf. Figure 1.2. If one of them goes bankrupt, the small size of the cross deposits will limit the effect on other banks. On the opposite, if banks hold deposits over a few (in the present case, just one bank) number of banks, the interbank market is "incomplete", cf. Figure 1.3. Banks will be more affected by the bankruptcy of one of them, possibly making them also defaulting. Obviously, if the network is such that some regions are not connected together (also called sparse network) cf. Figure 1.4, even through any intermediary region, a bankruptcy in one region can not affect the other region. To conclude, interbank markets where banks are connected together, but weakly, are more subject to contagion than other situations. In a related model, Allen and Gale (1998) explain how small shocks in a given sector may transform a banking crisis into a more widespread financial crisis. They do

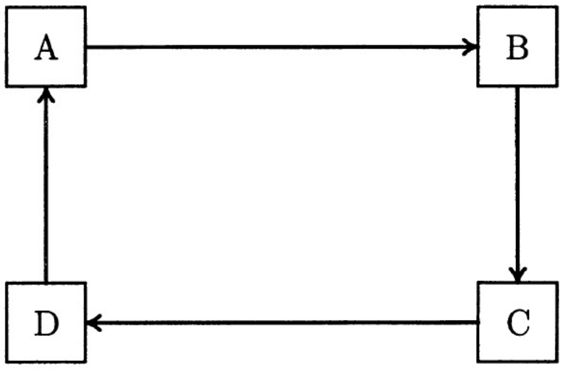

Figure 1.3: Incomplete network[a]

[a]Source: Allen and Gale (2000).

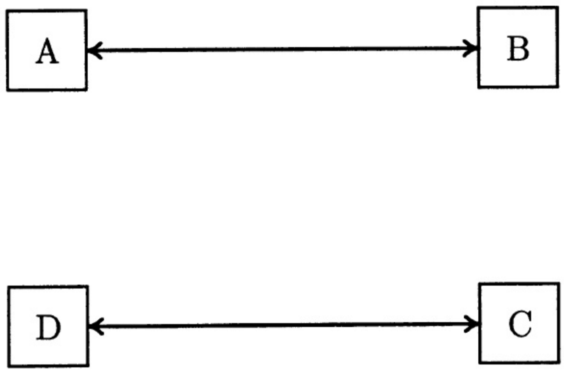

Figure 1.4: Sparse network[a]

[a]Source: Allen and Gale (2000).

not introduce shocks on liquidity demand or allow for cross holdings, but the "long" asset is risky, and cannot be liquidated at time 1. In this situation, bank runs are due to low-returns on the risky asset.

The framework introduced by Allen and Gale (2000) was extended in multiple ways. For example Dasgupta (2004) analyzes a two regions and three periods model, each region having one bank. There is one riskless asset, and each region also own a private long-term risky asset, with low short resale price. Depositors of each region also observe the depositors of the other region, especially the number of withdrawals. The authors suppose the senority of interbank holdings on individual deposits. Given this context, Dasgupta determines the thresholds of yield of the risky asset which lead to banks failures following regional liquidity shocks. He analyzes the failure of one bank conditional to the failure of the other one and exhibits an optimal level of interbank holdings depending on the probability of failure.

Leitner (2005) also distinguishes liquid and illiquid assets and the effects of the allocation of these assets over the banks. When liquid assets are too concentrated over a little number of banks, the whole network is threatened. This captures the risk associated with TBTF institutions.

1.2.2 Other major contributions

Generally speaking, to model financial connections, the authors introduce either direct connections representing assets from banks hold by other banks, or different types of assets that banks hold in a portfolio.

Freixas, Parigi, and Rochet (2000) analyze the same problem as Allen and Gale (2000). They study contagion in the interbank market following liquidity shocks. They prove that interbank holdings reduce banks' needs in liquid assets, and they enlight how credit lines (interbank cross-holdings) expose the system to gridlocks even if all banks are solvent, due to coordination mechanisms.

Eisenberg and Noe (2001) adopt a completely different approach of contagion effects. Indeed, they analyze the existence and the uniqueness of a "clearing payment vector" for complex financial networks, for example containing cyclical obligations. They determine how the system reacts to an idiosyncratic shock using this clearing payment vector. For example, they show that a shock on one bank lowers the value of holdings of all other connected banks. Their particular approach differs from the large majority of papers which exogenize the resale price of the shares of a defaulting bank.

Rotemberg (2009) determines the needs in liquid asset of firms that use it to settle their debts, depending on the structure of the network. He finds that when the number of links between firms increases, the optimal level of liquid asset also increases. He also analyzes the effect of financial concentration on debts' settlement process.

Morris (2000) calculates analytically the threshold of contagion between homogenous investors with special network configurations. His approach relies on the optimal behavior of agents depending on the others.

Egloff, Leippold, and Vanini (2007) also deal with contagion issues taking into account the impact of the credit portfolio's interdependence structure. Again, less diversified portfolios are more likely to create giant distress.

Cossin and Schellhorn (2007) analyze the effect of the network structure on firm's prices when default's risk depend on counterparties' defaults. Even if they prove that maximum diversification of counterparties reduces default risk in random networks, the optimal behavior in cyclical networks is to choose a finite number of counterparties to minimize risk.

Gale and Kariv (2007) prove that networks are likely to be incomplete given the cost (or lack) of information. When the network is incomplete, it is more vulnerable to shocks. The authors numerically illustrate this situation with a particular network composed of 5 agents.

Many questions about contagion have been studied through these models, such as the effect of the structure of the network (connectivity, concentration) on the risk of contagion, the resale price in case of default, the requirements in liquid assets, the coordination procedure. The network modelization of financial markets allows to deal with all these questions in a unique framework. It captures the common idea that intermediate connected banking systems are more vulnerable than highly or sparsely connected systems.

1.3 Random graphs

There are many applications of random graphs and networks theory in different fields of research, as illustrated by Newman, Strogatz, and Watts (2001) and Newman (2003). Especially, random graphs are subject to percolation phenomena, and this may be usefull to model financial contagion. A technical appendix at the end of the chapter gathers basic results of graph theory.

1.3.1 Percolation as contagion

Chapter 2 (our model) was motivated by the mathematical approach of percolation in random graphs. How could we transpose percolation to financial systems? The first step was the work of Watts (2002), who presents a simple model determining the probability of global cascades on a random network of interacting agents with a resistance effect (in finance this effect could be compared to a capital buffer). Precisely, Watts analyzes the probability and the size of the spillover starting from an initial "seed" in a network. The network approach is adapted to many different situations in physics, biology, but also economics. The network is represented by a random graph containing n agents. Agents can be represented by dots, or vertices, and links between them by edges. To get a concrete situation, we suppose that agents have to make choice 0 or 1 (buy or sell, trade or not, withdraw cash...) and their choices depend on the choices of agents they observe. When they observe an agent, there is an edge between the two dots representing the agents. These connected agents are called "neighbors". Among the neighbors of one agent, some are making choice 0 and some others are making choice 1. The current agent decides to make choice 1 if the fraction of his neighbors making choice 1 exceeds a private threshold drawn from an arbitrary distribution ϕ. Each agent is connected to k neighbors with a probability p_k. The average number of neighbors is z. Watts wants to determine the number of agents who adopt state 1, following the introduction of a fraction $f << 1$ of all the agents making choice 1 in the network. Unlike the complete network of Allen and Gale (2000), this model deals with incomplete networks, each agent is not connected to all the others, but has a limited number of neighbors. The incompleteness of the network is an important assumption: $\exists c > 0$ such that $z < c \ln(n)$. This is designed to guarantee the absence of local cycles of connections (Erdös and Rényi, 1960), which would highly change the behavior of agents, creating resistance effects. Indeed, suppose that 4 agents are linked consecutively, like Figure 1.3, one after the other. Suppose the first agent makes

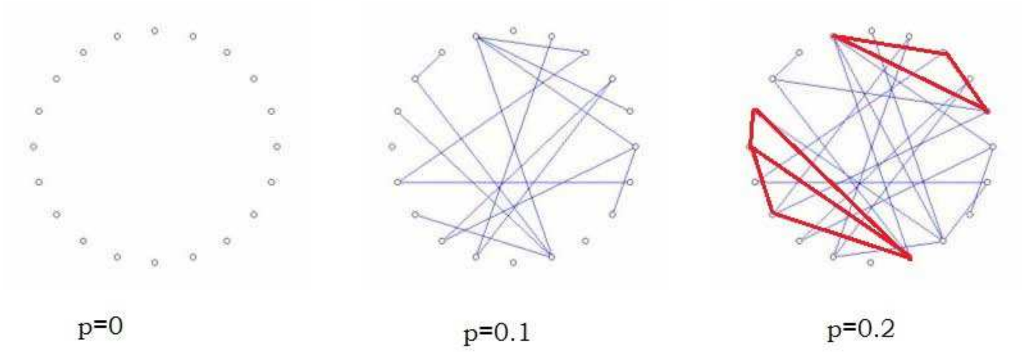

Figure 1.5: Example of uniform random graph $n = 20$. p is the probability to put an edge between any two points.

choice 1. The second agent is connected to 2 agents, the first and the third. As a consequence, if $\phi \leq \frac{1}{2}$, then the second agent adopts choice 1, and so on. On the contrary, let us suppose that agents 2, 3 and 4 are connected together and agent 2 is connected to agent 1. Agent 1 makes choice 1. Agent 2 adopts choice 1 if $\phi \leq \frac{1}{3}$. If $\phi > \frac{1}{3}$, then agent 2 chooses choice 0 and the contagion stops. On Figure 1.5, the probability to put an edge between any pair of agents p is identical for all agents (nodes). There are 20 agents. When $p = 0.1$, which represents on average 1 connection starting from half of agents, there are no cycles. On the opposite, when $p = 0.2$, which is equivalent to take $p = \frac{4}{n}$, there are local cycles, represented in red.

Watts determines the "early adopters" in the network: the immediate neighbors of the seeds, that adopt state 1. Let us consider an agent having k neighbors among which one seed (in state 1), and the other agents in state 0. This agent chooses state 1 with probability $\mathbb{P}(\phi \leq \frac{1}{k})$. From there Watts derives the generating function of an agent that has at least one neighbor in state 1: $G(x) = \sum_k p_k \mathbb{P}(\phi \leq \frac{1}{k}) x^k$. This function gives two important quantities: $G_0(1)$ determines the fraction of agents that may choose state 1 and $G_0'(1)$ the average number of connections made by these same agents. Then he extends the generating function to a set of connected agents (also called cluster) that might become 1 and he deduces the average size of such 1-clusters. The average size of 1-clusters diverges for some particular value of

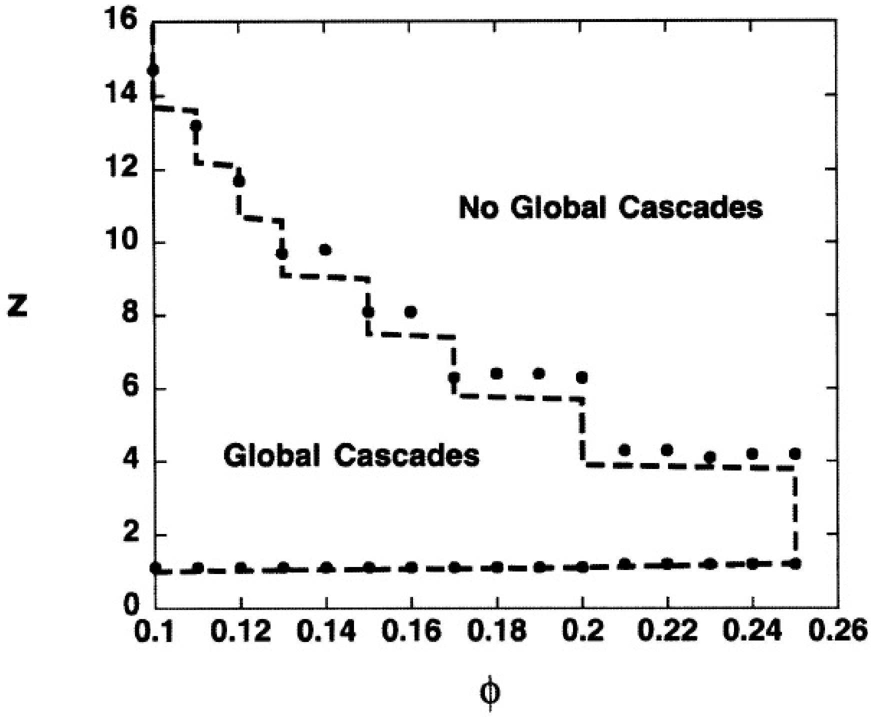

Figure 1.6: The global cascade window. The dash line is the theoretical result, and the solid circles represent numerical simulations. ϕ is the common threshold of decision and z is the connectivity parameter representing the average number of neighbors. [a]

[a]Source: Watts (2002).

the number of vertices, in the "phase" transition. Watts applies this reasoning to the particular uniform random graph, where the probability to put an edge between two agents is exactly $p = \frac{z}{n}$ and where the individuals thresholds of decisions are constant to the same ϕ^*. In this case, it is possible to determine theoretically (asymptotic properties when $n \to \infty$) and numerically (simulations for n large enough) the existence of a giant set of connected agents in state 1 within the network, depending on the value of z and of ϕ^*.

For $z < 1$ and $z > 15$ there are no global contagions in the economy and when $1 < z < 15$ they exist for any value of the individual threshold $\phi < 0.25$, as shown on Figure 1.6. In this case, the upper individual threshold ϕ^* such that giant 1-clusters exist is decreasing when z increases. Moreover, with the numerical simulations, the author calculates the frequency of these phenomena. He then extends the results to a larger class of random networks, without major differences in results. More connected networks are subject to larger contagions (cascades), but they happen less frequently. As the author explains, there is however a problem, random networks are not the best approximations for economic structures: agents are not acting purely randomly, they may choose their connections.

Percolation phenomena remain very interesting, because they capture the effect of the network structure on agents decisions. In the following, we study how these techniques have been closely adapted to financial networks.

1.3.2 Percolation as herding behaviors: size of the clusters

Another approach using the same percolation phenomenon is the work of Cont and Bouchaud (2000). They suppose that the dots (or vertices) represent traders. As soon as two traders are connected with a link, (or edge), they adopt the same behavior concerning an asset: buy, sell, or not trade. When the global size of the financial network $n \to \infty$; and when the probability to put a link between two agents is $p = \frac{z}{n}$ with $z = 1$, Bollobas (1985) proved that the size distribution of connected agents (or clusters) behaves as a power law. If $z \to_{<} 1$ the size distribution of the clusters behaves as a power law modulated by an exponential tail, which makes the variance finite. When $z \to_{<} 1$, the limit $n \to \infty$ proves that the number of neighbors of an agent is a Poisson random variable, with parameter z. They suppose that there is a fixed number of orders by unit of time. They also suppose that clusters are independant. There must be no trend in the action of agents, as a consequence, there exists a parameter $0 < a < \frac{1}{2}$ such that, for each cluster, $\mathbb{P}(\text{buy}) = \mathbb{P}(\text{sell}) = a$. The remaining case can be deduced: $\mathbb{P}(\text{not trade}) = 1 - 2a$. From there, they derive the law of excess demand for the asset, and the evolutions of the price of the asset. Precisely, their results prove that price changes have a heavy tail, and the "size" of the tail (variance) is inversely proportionnal to the number of orders per unit of time. This illustrates that illiquid markets are subject to larger price's variations.

We must enlight that contagion through imitation processes between traders have been explored through various approaches, and this influences the size of the tail of the return distribution. For example, Johansen, Ledoit, and Sornette (2000) introduce a “local self-reinforcing” imitation between noise traders. With a hazard rate, the noise between traders reaches a critical point such that all traders place the same selling order, which generates a crash. It is however optimal for traders to keep investing in the market out of crash periods, because bubbles[1] generate high profits before crashes. The resulting assets’ returns follow log-periodic power laws. The empirical study Johansen, Ledoit, and Sornette (1999) supports these distribution. Recently, this model was extended successfully to the recent Chinese stock market bubbles in Jiang, Zhou, Sornette, Woodard, Bastiaensen, and Cauwels (2010).

Focardi and Fabozzi (2004) make another contagion model using percolation. They also use the size of the clusters, when $z \to_{<} 1$, but instead of choosing the action (buy, sell, not trade) of the traders who constitute the cluster, they introduce a probability a of default on the cluster, which is the same for all clusters. The number of defaulting clusters Na is therefore proportionnal to N the number of clusters, and the number of defaulting traders $\frac{Na}{1-z}$ is proportionnal to the average size of the cluster $\frac{1}{1-z}$. Depending on how z is close to 1, using the power law cut off by an exponential law, they deduce a correlation vector of traders, due to the size of the clusters, and also deduce the distribution of aggregate losses. As they underline, there are some difficulties to use the theory of random graphs, because results hold for $n \to \infty$, but simulations help find interesting results.

1.3.3 Financial resilience to the test of percolation

Gai and Kapadia (2010) add some more financial features to Watts (2002). Indeed, they introduce a mechanism as the one of Allen and Gale (2000) in a random graph. In this model, the nodes represent banks. They consider oriented links instead of bilateral, this creates a directed graph. They add a weight on the links to represent the size of financial flows. As a consequence, they have to consider separately the

[1]A heuristic description of the rational bubbles of this model is in Chapter 3.

number of incoming links or liabilities L_i, to a bank i, also called the "in-degree" and the outgoing links or assets, the out-degree. They distinguish three types of assets:

- a liquid asset AL_i, which represents the interbank assets of other banks, hold by bank i. The interbank assets of a bank are supposed to be evenly distributed over the destinators;
- an illiquid asset A_i.
- deposits[2]: banks receive deposits from customers C_i. .

A bank is "solvent" if:

$$(1-f)AL_i + pA_i - L_i - C_i > 0, \tag{1.1}$$

where p is the price of the illiquid asset when sold; and f is the fraction of banks holding assets hold of bank i belonging to defaulting banks.

All banks are initially solvent and the authors shock the network by making one bank fail. The authors assume that when a bank fails, it does not refund any of its liabilities to other banks. As done in Watts (2002), they calculate the probability of default for a bank linked to the defaulting one. To calculate this probability, Gai and Kapadia assume that the capital buffer, namely $AL_i + A_i - L_i - C_i$, is a random variable (this implies that deposits are randomly distributed over banks). Obviously, when a bank has assets in a lot of other banks, the situation is less risky: because liabilities of one bank to the others are evenly distributed, the more counterparties, the lowest the amount. The default of one bank with a lot of counterparties will less impact the balance of his counterparties than the default of a bank with few counterparties. In the following, the authors determine the moment generating function of a neighboring bank, and the same for a cluster, using the property that the average in-degree is equal to the average out-degree. When the average degree is low, $z < 1$, there are no giant connected components (or clusters), and the default cannot spread over the whole network. When the degree

[2]We adopt notations similar to our's to make it simpler.

increases: $z > 1$, there are larger components until there are giant components in the network. In this case, larger clusters may default (systemic risk), but since banks assets are distributed over more counterparties, this lowers the risk of default. As a consequence, the more connected the network, the more resistant to shocks. When the network is dense, it is more resilient to shocks, but when contagion spreads, the whole network is threatened, this creates large failures. When the degree is too high, there are too many banks with a lot of counterparties, and there is no contagion anymore. By simulation over smaller networks, they get very similar results to Watts (2002). A very close model was designed by Amini, Cont, and Minca (2010).

Another analysis of networks is done by Nier, Yang, Yorulmazer, and Alentorn (2007). They also use a random network structure. An interesting contribution of this study is the calibration of the parameters of the model: the number of banks 25, the probability to make a link between two banks 0.2, the weight of the links, the percentage of interbank assets 20%, the percentage of capitalization (cash of the bank) 5%. They get related results: more capitalized banks better withstand defaults. Increasing connectivity first increases risk and then helps the network to resist contagion. When the size of interbank liabilities increases defaults locally spread more easily.

Gai, Haldane, and Kapadia (2011) analyze a network made of banks having different types of assets: secured (safe) and unsecured (risky) interbank liabilities, deposits, and capital. As in Gai and Kapadia (2010), links are directed, banks have assets and liabilities. When needing liquidity, the secured assets keep their whole value, the others are subject to “haircuts”, which also represent the risk associated to them. In addition, each bank is also subject to an additional specific haircut that decreases the value of his own unsecured assets, this represents idiosyncratic shocks. Following a shock, one bank has liquidity problems. To avoid defaults on payments, this bank needs to increase liquidity. To do that, the authors assume that this bank will try first to sell unsecured interbank assets. The contagion mechanism is as follows, when a bank is subject to a “hoarding” action from other banks, it looses a part of his liabilities. This is likely to create liquidity problems in that bank, which may in response start to withdraw his deposits in other banks. Using this mechanism and simulations, the authors determine the contagion over the network following a shock on one bank. The results highly depend on the value of parameters, among which, the resale price of the unsecured asset, the complexity of the network (number of connections) and the different fractions of types of liabilities. Of course, lowering the resale price of unsecured assets increases contagion. Given the number of links, when the network is more concentrated over a few major banks, this increases the risk. Given the concentration, more complex networks also increase risk. Their results recommend to strengthen liquidity requirements.

The impact of actions of the agents on the network has been analyzed in Anand et al. (2012). They mix random networks and global games as done by Morris and Shin (2003). The banks who constitute the network have two types of assets: interbank assets A_i, private equity (composed of cash) C_i, and liabilities L_i. Creditors of each bank receive information about the bank. Actually each creditor is another bank in the network. Using the information, they decide, at each period, what to do about the bank: continue to lend, or foreclose their funds. The foreclosure leads to 0 payoff, whatever the further state of the bank. A bank may remain solvent or become defaulting, depending on the decisions of creditors. When a creditor decides to rollover a loan, the payoff depends on the state of the bank. When the bank is solvent, the creditor j receives $(1 - a_j)$ and when the bank defaults, the creditor receives a negative payoff $-a_j$. Depending on the value of a_j, three situations arise. In the extreme cases where $a_j < 0$ or $a_j > 1$, the game has a unique Nash equilibrium. For intermediary situations $0 < a_j < 1$, there is no dominant strategy, between foreclose and rollover: there are multiple equilibria. The payoff of the creditor is influenced by the other creditors: the more creditors rollover, the lower a_j, and on the opposite, the more creditors foreclose, the highest the risk of default from the bank. To solve the problem, the authors assume that a_j is a random variable common to all banks, that they finally fix to a constant a. Using global games theory, the unique Bayes-Nash equilibrium is a threshold strategy. Imposing Laplacian beliefs about the actions of the other creditors[3], they

[3] The fraction of creditors who rollover is a discrete uniform variable on $[0, 1]$.

prove that there is a unique equilibirum: when the creditor j specific parameter a_j satisfies the relation: $a_j \leq \frac{A_i+C_i}{L_i}$, then the creditor rollovers the loan of bank i. In the other case, he forecloses. The time evolution of the game is as follows. At the dates they receive information, following a Poisson process of parameter t, the creditors meet together and decide to rollover or to foreclose their credits. Credits are initially created at Poisson process times of parameter "*ini*", and between random banks in the network. Existing credits also mature at another Poisson rate "*end*". The authors deduce the stationnary state of the complete process, and the state of the network, depending on a_j, and the 3 other time parameters: t, *ini* and *end*. To establish the results, they assume that all creditors (all banks) have the same individual threshold of the game rollover/foreclose $a_j = a$, and all banks have the same level of cash $C_i = C$.

The results are intuitive. When the threshold a is low, the network is dense, creditors rollover their loans. For high a, the state of the network relies on the amount of cash C of the banks. As a consequence, when the ratio asset-liabilities decreases, the network is prone to contagion of foreclosures and contagion of defaults. When the debt maturity is increased (this correspond to a decrease of the debt maturation parameter *end*), the network can be dense, if a is low, or sparse in the other case.

For intermediary values of the foreclose parameter a, a hysteresis phenomenon takes place, the connectivity of the network slowly increases until reaching a highly connected state which leads to a gobal foreclosure strategy, and creates a low connected network. The authors also deduce an endogenous rate of bank failure, which has two solutions for the same intermediary values of the time parameters. This rate can be either close to 0 or strictly positive. As a consequence, two different situations exist: almost no defaults or a net positive rate of failure. The authors deduce policy implications to improve the state of the economy: increasing the capitalization of agents reduces foreclosures; and transparency (or increasing the frequency of information) avoids hysteresis phenomena. This model of random graphs and global games is very relevant, because it models the effects of the agents on the network – foreclose or rollover – and the effect of the network on agents' decisions – mainly through threshold effects. It also introduces a time dynamic in random networks, which is innovative. This is done at the cost of a hard mathematical tractability (master equation of the generating function...) and new assumptions, especially the one on the random time rate creation of links. The authors introduce nice mechanisms for example, the payoff of the creditor, but, to get results, they have to adopt special values common to all agents instead of individual random variables. This is a bit regrettable: without clearly explaining, the model seems to be general while it becomes more "specialized".

1.3.4 General remarks on financial networks

A technical constraint of the literature of random graphs must be enlighted. A majority of authors wants to use the percolation phenomenon (Watts (2002), Anand et al. (2012), Gai et al. (2011), Gai and Kapadia (2010), Cont and Bouchaud (2000), Focardi and Fabozzi (2004)) in classical or oriented graphs. They all require the probability to make a connection (link, debt, asset...) to be close to the percolation threshold: $\mathbb{P}(\text{link}) = \frac{\text{constant}}{n}$, when there are n agents. This limits the scope of the models. In the same field, to keep the random aspect of connections – otherwise the mathematical literature would not apply – unusual hypothesis or mechanisms are adopted, such as the random creation of links, the binary transmission of choices, or clearing mechanisms. Purely random connections can be justified in biology or chemistry, because when used to deal with microscopic structure, the molecule or virus's attacks, or the percolation in porous materials are precisely random. However, assuming that financial links between agents are purely randomly created remains surprising. We propose in the next chapter a model to avoid those two problems.

Many remarks of the literature explain that even with a low number of agents ≈ 500 and an intermediate number of simulations ≈ 1000, the results of simulations are consistent with the theoretical results. The same reasoning is applied in chapter 2, and the simulations are considered as truthful. This could also support the fact that real financial networks have a finite number of agents, which is of the same order of magnitude as in Bech and Atalay (2010). In micro approaches of networks models, Hale (2011) consider 7938 banks over 141 countries. In other studies, the financial flows are aggregated by countries, like Chinazzi, Fagiolo, Reyes, and Schiavo (2013) or Minoiu and Reyes (2013).

We use the same concepts of network theory as the aforementionned literature, we present in the following chapter a macro network based on the behavior of agents, we determine by an algorythm the state of the network depending on agents' choices. We do not use the same formulation for links between agents and we introduce a different resistance/contagion mechanism along financial links. Instead of studying the effect of an exogenous shock, such that the failure of a bank, we endogenize the formation of the failure within the network. As Anand et al. (2012) (and Allen and Gale (2000) for the interbank holdings), we use a "zero recovery assumption", when a bank defaults. As Nier et al. (2007) and according to Bech and Atalay (2010), we allow the number of links to exceed the percolation threshold. We question about the efficiency of capitalization ratio. We especially end by studying two unusual questions: is the ratio independant of the absolute level of capitalization? When increasing the ratio with a fixed number of links, the systemic risk decreases. What is the effect of increasing the ratio when the absolute capitalization is constant?

Appendix A

Sandard results of graph theory

Definitions

We consider N points, also called nodes or vertices $(V_1, V_2, ..., V_N)$. An edge, (link or connection) is a pair of vertices (i, j). A graph is a set of vertices V and a set of edges E. Any pair of vertices can be linked by an edge. Given the graph $\mathcal{G}$, two vertices V_1 and V_2 are "neighbors" if the edge $(V_1, V_2) \in E$. Given a vertice V_1 in the graph, the degree of this node is the number of its neighbors in the graph.

$$deg(V_1) = \# \{V' \in \mathcal{G} : (V_1, V') \in E\} . \tag{A.1}$$

A path between two nodes V^* and V' is a finite sequence of vertices:

$$V^* = V_0, V_1, ..., V_{n-1}, V_n = V',$$

such that for any $k \in [0, n-1]$, the edge $(V_k, V_{k+1}) \in E$. A graph is "connected" when any pair of nodes can be linked by a path. A cycle is a path such that the extermities coincide. A graph is a "tree" if it is connected and none of its subgraphs contains a cycle. A subgraph is a graph with a set of vertices V_s and a set of edges E_s such that $V_s \subseteq V$ and $E_s \subseteq E$.

Random graphs are such that for any pair of nodes there is an independant random probability p to put an edge between these two nodes. Such a graph is noted $\mathcal{G}(N, p)$. There is therefore a probability $(1-p)$ to not have an edge between two vertices. A vertice is degree k if it has exactly k neighbors. This gives the probability law of the degree. Let p_k be the probability to have exactly k neighbors.

$$p_k = \binom{N-1}{k} p^k (1-p)^{N-1-k}. \tag{A.2}$$

When $N \to \infty$ we deduce that $p_k \sim_{N\to\infty} \frac{z^k \exp -z}{k!}$ with $z = p(N-1) \sim_{N\to\infty} pN$ is the average degree of the vertices. This means that the degree distribution behaves asymptotically as a Poisson distribution with parameter $z = pN$.

Arbitrary degree distribution

Newman (2003) focuses on random graphs with arbitrary distribution degree. This represents the sets of graphs such that $\forall X$ vertice of V, the degree probability is: $\mathbb{P}[deg(X) = k] = p_k$, where $(p_k)_{k\leq 0}$ is given. Erdös-Rényi graph corresponds to taking for $(p_k)_{k\leq 0}$ the Poisson distribution with parameter $z = pN$.

The generating function of a discrete random variable X is the power series expansion, which converges on $[-1, 1]$:

$$G_X(x) = \sum_{k=0}^{\infty} \mathbb{P}[X = k] x^k. \tag{A.3}$$

Let $G_X^{(i)}$ the ith derivative of G_X. Then

$$k! p_k = G_X^{(k)}(0). \tag{A.4}$$

The ith moment $\mu_i = \mathbb{E}\left[X^i\right]$ is related to the generating function by:

$$\mu_i = \left(x \frac{d}{dx}\right)^{(i)} G_X(x) \Bigg|_{x=1}, \tag{A.5}$$

where $\left(x\frac{d}{dx}\right)^{(i)}$ means differentiate w.r.t. x and multiply by x iterated i times. We especially have $\mathbb{E}[X] = G'_X(1)$.

Now we can calculate the size of the connected components (also called clusters), using the method of Feld (1991) and Newman (2003). Consider a random edge, and one of the two vertices joined by this edge. This vertice has a number of neighbors. The generating function of the distribution of the number of neighbors of this vertice is proportional to the degree of the vertice times the probability of the vertice to have this precise degree: kp_k, . The distribution probability must be normalized, because the sum of the probabilities must be 1. As a consequence, the probability to reach a neighbor having exactly k neighbors is:

$$\frac{kp_k}{\sum_{i=1}^{\infty} ip_i} = \frac{kp_k}{\sum_{i=1}^{N} ip_i}. \tag{A.6}$$

We deduce the exact generating function of the distribution of the number of neighbors of this vertice and the expression in terms of G_0:

$$\frac{\sum_{k=1}^{\infty} kp_k x^k}{\sum_{k=1}^{\infty} kp_k} = \frac{xG_0'(x)}{G_0'(1)}. \tag{A.7}$$

The next step is to determine the number of neighbors of one neighbor (who are also called the second neighbors). However, in this probability, we must not include the first vertice, because it is also a neighbor of the first neighbor. The probability to reach k “second neighbors” is proportionnal to kp_k. To normalize the distribution, we get:

$$q_{k-1} = \frac{kp_k}{\sum_i ip_i}. \tag{A.8}$$

The generating function of the distribution is: $G_1(x) = \sum_{k=0}^{\infty} q_k x^k$.

$$G_1(x) = \frac{G_0'(x)}{G_0'(1)}. \tag{A.9}$$

In this formulation, there is one problem, there is a positive probability that a neighbor of a vertice A has a neighbor which is also a direct neighbor of A. This would create a cycle, and this common neighbor would account for two neighbors instead of one. Fortunately, as proved by Newman (2003), the probability of a vertice belonging to a cyclic component tends to 0 when[1] $N \to \infty$ as $N^{\frac{-1}{3}}$. Thus we can consider that there are only trees in the graph.

The generating function of the sum of two independant random variables is the product of the generating functions of the two variables. As a consequence, the generating function of the sum of the neighbors of k neighbors is $G_1(x)^k$.

Starting from a random edge, we reach one of the two vertices. From this vertice, we look for all the other vertices that are connected. This set (or cluster) has a number of vertices. Let $H_1(x)$ be the generating function of the number of vertices of the clusters. The number of neighbors of the initial vertice has the distribution function $G_1(x)$. If the initial vertice reaches m neighbors, we can also consider the distribution of the m clusters generated by these m neighbors. The distribution of the size of the m clusters reached by the initial vertice is $H_1(x)^m$. The total number of vertices that the initial vertice can generate is therefore:

$$H_1(x) = x \sum_{k=0}^{\infty} q_k (H_1(x))^k = xG_1(H_1(x)). \tag{A.10}$$

The multiplier term x comes from the fact that our initial vertice at the end of the random edge is connected to another vertice, which makes a change in the index, x^{k+1} instead of x^k. A similar way, we could also calculate the second neighbors of an agent: $G_0(G_1(x))$, and so on. To get the distribution size of a whole cluster to which a random vertice belongs, we introduce the distribution G_0:

$$H_0(x) = x \sum_{k=0}^{\infty} p_k H_1(x)^k = xG_0(H_1(x)). \tag{A.11}$$

Solving the equation (A.10) leads to the solution of (A.11). Then we can deduce the average size of the clusters: $H_0'(1)$:

$$H_0'(1) = 1 + G_0'(1)H_1'(1). \tag{A.12}$$

[1]This is true as $z < 1$.

Because (by definition of generating function) $G_0(1) = H_1(1) = 1$, and using equation (A.10) applied at $x = 1$, we get:

$$H_1'(1) = \frac{1}{1 - G_1'(x)}; \tag{A.13}$$

which is used in equation (A.12) to get:

$$H_0'(1) = 1 + \frac{G_0'(1)}{1 - G_1'(1)}. \tag{A.14}$$

$G_0'(1)$ represents the average degree z (or number of neighbors) of any vertice. $G_1'(1)$ is the average number of neighbors of one neighbor. $G_0'(1)G_1'(1)$ is the average total number of second neighbors of one agent z_2. The average cluster size is $1 + \frac{z^2}{z - z_2}$. The size diverges when the average number of second neighbors is the same as the average number of neighbors, or $G_1'(1) = 1$. This is called the phase transition.

For the standard Erdös Rényi model, $G_0(x) = \exp z(x-1)$, where $z = pN$. $G_1(x) = G_0(x)$.

Percolation in Erdös Rényi model

Authors are interested by properties of random graphs when p the probability to put an edge is close to $\frac{z}{N}$, where z is a constant. Indeed, the main results come from the behavior of the limit of $\mathcal{G}(N, \frac{z}{N})$ when $N \to \infty$. The distribution of cluster size can be well approximated using the moment generating function. As previously done, we neglect the possibility of cycles in the components. $C(k)$ represent the probability of a vertice to belong to a cluster of size k when $n \to \infty$. ($C_N(k)$ is the probability of a vertice to belong to a cluster of size k in a graph of N vertices.) The moment generating function is:

$$\phi(x) = \sum_{k=1}^{\infty} \exp(kx) C(k) \tag{A.15}$$

$\phi_N(x)$ is the moment generating function in a graph of size N. As Amini et al. (2010), we start from a graph of size N and we add one vertice. The probability to create m new edges is $\binom{N}{m}\left(\frac{z}{N}\right)^m\left(1-\frac{z}{N}\right)^{N-m}$. Because we suppose that there are only trees and no cycles, the m new edges all reach different clusters, to avoid creating a cycle. Let be $c_1, c_2, ..., c_m$ the size of these different clusters. The new created cluster is of size $1+\sum_{i=1}^{m} C_i$. This process is described on Figure A.1.

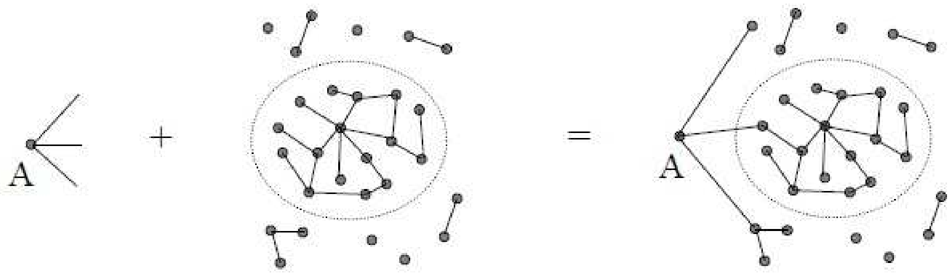

Figure A.1: Construction of a cluster of size $N+1$[a].

[a]Source: http://www.phys.ens.fr/ monasson/

$$C_{N+1}(k) = \sum_{m=1}^{N} \sum_{c_1,c_2,...,c_m} \binom{N}{m}\left(\frac{z}{N}\right)^m \left(1-\frac{z}{N}\right)^{N-m} \delta(c_1+c_2+...+c_m+1-k)$$
$$C_N(c_1)C_N(c_2)...C_N(c_m). \qquad (A.16)$$

We deduce the relation between the generating functions:

$$\phi_{N+1}(x) = \exp x\left(1+\frac{z}{N}+\phi_N(x)\frac{z}{N}\right)^N. \qquad (A.17)$$

When $N \to \infty$ the limit is:

$$\phi(x) = \exp x + z(\phi(x)-1). \qquad (A.18)$$

Differenciating this equation n times and applied in $x=0$ yields the n-th cumulant. Using the recursive relation between cumulants and moments, the authors deduce that the distribution of clusters sizes $H(k)$ is:

$$H(k) = f(z)\frac{C(k)}{k}; \qquad (A.19)$$

where $f(z)$ is a normalization constant.

Erdös and Rényi (1960) proved that sizes of largest components change according to the parameter z:

- when $z < 1$ the largest clusters have a number of vertices which is of magnitude:
$$\frac{\ln(N) - \frac{5}{2}\ln(\ln(N))}{z - 1 - ln(z)}.$$
The number of such components is finite when $N \to \infty$. When $z \to_< 1$ the probability density for the cluster size distribution decreases as a power law modulated by an exponential tail. $\exists a, k_0$ such that
$$P(k) \sim_{k\to\infty} \frac{a}{k^{\frac{5}{2}}} \exp\frac{(1-z)k}{k_0}.$$
The avergae cluster size is $\frac{1}{1-z}$.

- when $z = 1$ there exist two positive constraints b_1 and b_2 such that the size H_i of the i largest clusters satisfies:
$$b_1 N^{\frac{2}{3}} < H_i < b_2 N^{\frac{2}{3}}.$$
The number of these clusters is finite when $N \to \infty$. The probability density for the cluster size distribution decreases as a power law:
$$P(k) =\sim_{k\to\infty} \frac{a}{k^{\frac{5}{2}}}.$$

- when $z > 1$, there is one largest cluster, and its size is a lot larger than the others. The size H of the largest cluster is:
$$H = Ng(z),$$
where $g(z)$ is the unique solution to the equation $1 - g = \exp(-zg)$.

The particular phenomenon happening for $z = 1$ is the percolation, and it corresponds to the union of all largest clusters to create a giant component. To

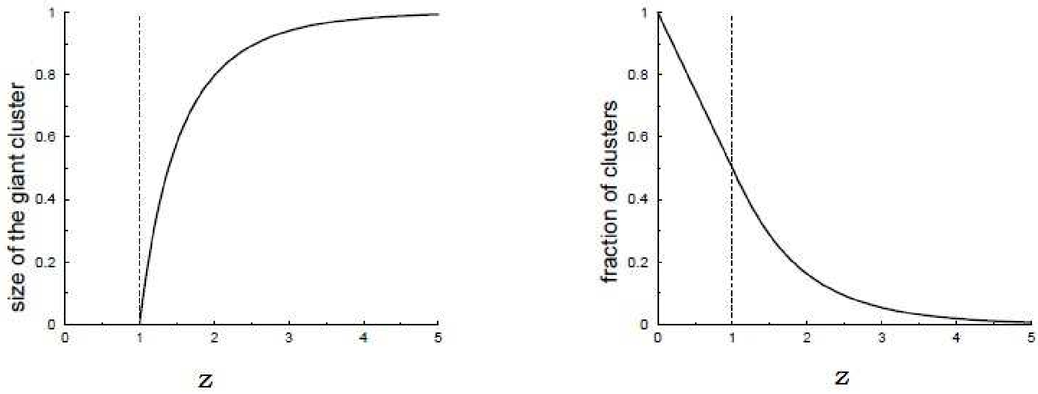

Figure A.2: Size of the largest cluster and fraction of cluster by vertice as a function of the parameter z.[a]

[a]Source: http://www.phys.ens.fr/ monasson/

illustrate these results, Figure A.2 shows the size of the largest cluster and the fraction of the number of clusters on the number of dots in the graph.

Directed graphs

In directed graphs, edges have a direction, from one vertice to the other. Each vertice of the graph has therefore two degrees, an "in-degree" and an "out-degree", which respectively are the number of edges pointing into the vertice, and out of the vertice. There are two degree distributions, or a joint degree distribution: p_{ij} represents the the probability that a vertice has in-degree i and out-degree j. This method is developed by Newman (2003). The same way, the "in-component" is the set of vertices that can reach a given vertice, and the "out-component" is the set of vertices that can be reached from a given vertice. The generating function of the degree of a vertice in the directed graph is:

$$G(x, y) = \sum_{i,j=0}^{\infty} p_{ij} x^i y^j. \tag{A.20}$$

The means of the in-degree and the out-degree are given by the first partial derivatives of G w.r.t. x and y. Because every edge pointing into a vertice is also starting from another vertice, these two means are equal. This value is called the mean degree z. The generating function must therefore satisfy:

$$z = \left.\frac{\partial G}{\partial x}\right|_{(x,y)=1} = \left.\frac{\partial G}{\partial y}\right|_{(x,y)=1}. \tag{A.21}$$

This relation also imposes a constraint on the probability distribution:

$$\sum_{ij}(i-j)p_{ij} = 0. \tag{A.22}$$

As in undirected graphs, it is possible to define generating functions of the number of neighbors reached by a random vertice $L_0(x)$, and of the number of vertices leaving a neighbor reached by a random edge $L_1(x)$, and their corresponding equivalent generating functions for incoming edges. $M_0(x)$ and $M_1(x)$. We can write the following relations between all generating functions:

$$\begin{aligned}
L_0(x) &= G(x,1); \\
L_1(x) &= \frac{1}{z}\left.\frac{\partial G}{\partial x}\right| y = 1; \\
M_0(x) &= G(1,y); \\
M_1(x) &= \frac{1}{z}\left.\frac{\partial G}{\partial y}\right| x = 1.
\end{aligned}$$

As made for undirected graphs, it is also possible to determine the size of the clusters. The equivalent equation of (A.14) is $M_1'(1) = 1$. This gives the relation:

$$\sum_{i,j=1}^{\infty}(2ij - i - j)p_{ij} = 0. \tag{A.23}$$

Chapter 2

Systemic risk and capitalization ratio in an homogenous financial network[1]

Contents

[1]The current chapter is a joint work with Camille Cornand and Mehdi Senouci. Previous versions have been presented at: EEA 2009 conference, AFFI 2009 conference, CEFAG 2010 seminar.

2.1 Introduction

2.1.1 Motivations

At the root of this piece of research, like many authors, we had the idea to see how financial distress precolates through a financial network. The idea of percolation is associated with two phases of the network: one is such that agents are not connected enough, and defaults remain locals. The other one is such that financial links allow defaults to propagate all over the network. The first phase appears at the left of some critical number of links, while the second one appears at the right of it.

For the transmission of defaults, authors have been adding various resistance mechanisms of the agents, as we saw in the last chapter. However, this percolation phenomenon takes place when the probability to put an edge between any two agents is of type $\frac{z}{n}$ with $z \approx 1$. One contribution of this work is to relax this restrictive assumption and uncover an interesting second regime. In addition, adding random links in a network is also a very specific assumption. But still we believe that the network representation of the financial system is relevant.

In our model, we do not consider purely random networks. We suppose that financial links are built to satisfy conditions on the balance sheets of the agents. We mainly focus on financial intermerdiaries such as banks, funds, insurance companies. For example, if an agent has a large amount of assets, we suppose that he must also have a large amount of liabilities. We allow the number of financial connections to widely exceed the percolation threshold of the network. Moreover, we consider a large but finite number of agents. We do not investigate into the asymptotic properties of the network. We want to reproduce two distinguishing features of recent financial systems:

- the network has an influence on the financial entities. For example, what is the effect of the defaulting bank on its counterparties, and on the counterparties of the counterparties?

- The financial entities build and shape the financial network, for example, by choosing their counterparties.

Using such a network, we want to capture the high level of financial connections, the possibility of systemic risk and determine the role of the regulator: increasing the capitalization ratio forces the agents to increase the capital, which makes them more resilient to the transmission of financial distresses, but it also reduces the number of financial links in the economy, which may decrease the risk sharing.

2.1.2 Overview of the model

We consider a network populated by a large (but finite) number n of agents owning the same amount[2] of capital C. We limit the amounts of their assets by a capitalization ratio ϕ.

We suppose that agents make investments towards other agents. All investments have the same financial size, that we normalize to 1. An investment lasts one period. When it matures, the investment is supposed to be repaid to the issuer. In this case, the issuer has a net positive payoff e. Alternatively, if the investment is not repaid, the issuer of the investment loses the value of his investment.

Financial agents are infinitely lived and optimize the discounted sum of their expected payoffs. At each period, they choose the number of investments they issue, and the destinators of their investments. This determines the structure of the financial network.

Shocks on the network are endogenous, because they correspond to strategically defaulting agents: when some agents are destinators of a large number of investments, and issue less investments, they can choose to default. In this case, they keep the received investment (or liabilities) and they give up their own investments (or assets). Giving up one's investments somehow corresponds to a fine.

Because of these stragically defaulting agents, some issuers lose their investments, and so might also default, because their liabilities exceed their remaining

[2]The model does not focus on concentration phenomenons, like TBTF, which would require a small number of bigger agents.

assets added to their capitalizations. In this case we assume that these agents are liquidated and their assets lose their entire value. Depending on the maximization horizon of agents, short (one period), long (infinite periods) or myopic[3] (intermerdiary cases), we derive the number of financial links of the network, the number of strategically defaulting and contaminated defaulting agents, and the payoffs of each type of agent. We analyze the role of the regulation on the limitations of the number of investments. We prove theoretically that well-defined solutions to the problem exist. Then we simulate the solution using matlab, and we state our results.

2.1.3 Implications and results

Short-horizon maximizations lead to less strategically defaulting agents than long-horizon maximizations. This result corroborates the intuition. The network converges to a dense state when agents optimize in the long-run and sparse state when agents optimize over one period. However, for myopic behaviors, the state of the network of agents depends on their capitalization: for highly capitalized agents, the optimal choice is to create a dense network, which includes a non-negligible systemic risk: all agents are defaulting. Limiting the assets of agents by a prudential ratio decreases the systemic risk but does not completely eliminate it. For low capitalized agents, such a prudential ratio may have counterproductive effects, because it may create a sparse network, and dry up the liquidity.

Section 2 lays the model of the network as a matrix and details how some agents enter strategic default while others do not. In Section 3 we analyze the investment decisions of financial agents as functions of their expected payoffs. Section 4 presents the results of the simulation of the network in short-term optimisation. Section 5 extends the results to all time-horizons, and delivers the policy implications. Appendix B describes the heuristic way of computing simulations of the model.

[3]loosely speaking: a finite number of periods.

2.2 Modelling the network

2.2.1 Investments, financial connections

The financial system is a network populated by a large number n of agents or financial intermediaries (traders, investments banks, insurance companies...). To simplify, all agents are supposed to be rational and risk neutral[4]. We assume that agents are homogenous: they all have the same endowment, also called capitalization. Agents are infinitely lived and time is discrete. In the model, an "investment" is a financial contract.

Financial intermediaries make investments of the same financial size towards other financial intermediaries, at each period, expecting some positive payoffs. This constitutes the financial network. Investments last one period. At the beginning of each period, agents simultaneously choose their targets for investments and the number of their investments. These investments are supposed to be paid back at the end of the period: this is the clearing process. Because investments change at each period, there is a new realization of the financial network at each period. As in Anand et al. (2012), the agents' investment choices are not purely random. We assume that agents know the number of investments planned by the other agents. This means that agents have an accurate estimation of the number of assets of other agents, while they ignore their number of liabilities. This catches the difficulty to evaluate the assets-liabilities ratio of the agents and models counterparties risks. In other words, agents do not know the exact structure of the network. They choose the number and the destinators of their own investments to maximize their expected payoffs. When an agent makes an investment towards any other agent, he expects to get back his investment at the end of the period, with a net positive yield.

A graph representation of the financial network is possible, as in Allen and Gale (2000) or Gai and Kapadia (2010). Each agent is represented by a dot. Financial flows between agents are represented by oriented edges between the different dots.

[4]The results still hold for risk-averse agents, as the concavity of the utility function would not impact results to a large extent.

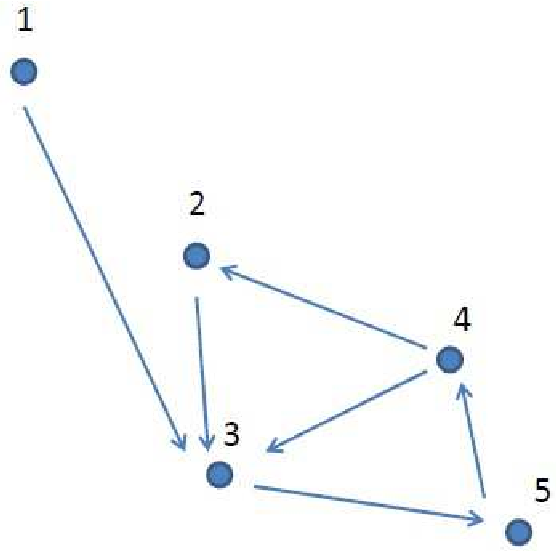

Figure 2.1: Example of financial network with 5 agents

The directions of the arrows represent the claims on investments, i.e. the directions of the financial flows at the end of each period. An example of financial network is provided on Figure 2.1.

As all the outgoing financial flows and incoming financial flows of any dot have the same standard financial size[5], all financial flows can be represented by a $(n \times n)$ matrix M between agents. For example, to model an investment from agent i to agent j, we add an oriented edge on the graph from agent j to agent i, and we add in the matrix the number 1 on line i and column j: $M_{ij} = 1$. Any investment from j to i is a debt from i to j, so the matrix representing the links between agents is anti-symmetric $M_{ji} = -1$. We suppose that only one investment is possible between any two agents, so that multiple and reciprocal investment do not exist. Because investments are reallocated at each period, the matrix also changes. At time t, the matrix is denoted by M^t. For example, the matrix corresponding to the previous graph on Figure 2.1 is:

[5]We normalize the values of all financial flows to 1.

$$M = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 & -1 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}. \tag{2.1}$$

2.2.2 Structure and allocation of investments

Agents expect to make profitable investments: there is an arbitrary payoff e such that any refunded investment is supposed to return some gain. By contrast, an investment that is not paid back is definitively lost, and costs one unit (the amount of the investment). From a financial point of view, the parameter e may represent the real interest rate on a financial contract on the interbank market, for example.

At each period t, each financial agent i receives some number of investments also called liabilities[6], L_i^t and makes a certain number of investments (also called assets), noted A_i^t. These two numbers can be expressed using the matrix of the financial links:

$$A_i^t = \sum_j (M_{ji}^t)^+, \tag{2.2}$$

$$L_i^t = -\sum_j (M_{ij}^t)^-, \tag{2.3}$$

where $.^+ = \max(0, .)$ and $.^- = \max(0, -.)$.

At the end of the period, any agent has to refund the investments he received, and his investments are also redeemed. We assume that the investments of each financial agent A_i^t are limited to a fraction of the capitalization[7] C by a prudential debt to capital ratio ϕ fixed by the regulator, such that:

$$\boxed{\phi A_i^t \leq C.} \tag{2.4}$$

[6] An agent has no choice but to accept an investment from any other agent.

[7] Recall that all agents are equally capitalized: $\forall (i, j), C_i = C_j = C$. In addition, we suppose that capitalization is constant over time.

This constraint captures the usual capital requirements. For each agent, the following relation between his assets A_i and his liabilities L_i must be satisfied, illustrating the balance sheet:

$$\boxed{C + L_i^t \geq A_i^t. \qquad (2.5)}$$

When $C + L_i^t - A_i^t > 0$, there is some cash that is not invested by agent i.

To summarize, at each period, financial agents make investments given the number of investments of other agents. The allocation of investments is restricted to the networks, such that, for each agent, both conditions (2.4) and (2.5) are satisfied.

2.2.3 Contagion mechanism

Any agent expects to get back his investments A_i^t at the end of the period. Some of them might not be actually paid back as we shall see later. Let AR_i^t be the investments refunded to agent i. If $AR_i^t + C < L_i^t$, agent i does not have enough cash to refund all his liabilities. In this case, the agent is said to be "contaminated defaulting". He does not refund any of his creditors, because we assume that the liquidation value of his assets is zero. Like Allen and Gale (2000) and Anand et al. (2012), we use a "zero recovery assumption" when there is a default[8].

On the opposite, any agent i such that $AR_i^t + C > L_i^t$ refunds all his creditors, and is said to be "healthy". To calculate the exact number of defaulting agents, we study the contamination of the defaults along the paths of investments also called lines of credit. As explained by Watts (2002), paths may have different forms that account for their resistance: trees are more vulnerable than cycles.

The higher the capitalization C, the easier it is for an agent to resist default on an investment. From a macroeconomic point of view, the prudential ratio also represents the capacity of the network to limit the proliferation of financial distress along credit chains, as will be proved later on.

[8]This was a major feature of the recent crisis. Relaxing this assumption, for example, refunding a random part of the creditors would generate significantly less defaulting agents.

2.3 Strategic default and agents' choices

Within the financial network, given agents' investments, at time t, some agents are net creditors: $A_i^t > L_i^t$ (they make more investments than they receive) and some other agents are net debtors $L_i^t > A_i^t$. The net debtors might be interested in strategically defaulting and quiting the network during the period: they will not pay back the investments they received at the beginning of the period. They strategically default if the gain of this action exceeds the discounted sum of their expected payoffs from investments. In this case, they definitively leave the financial network[9]. A strategically defaulting agent leaves the network before the end of the period and therefore also gives up his own investments. The corresponding fine associated to this behavior is a foreclosure of assets. This form of default allows to endogenize the formation of defaults, while most of the literature concentrates on the effect of an exogenous shock[10] on the financial network. Making a strategic default, because the gain of default exceeds the fine, consisting in giving up one's investments was introduced in general equilibrium theory with incomplete markets by Dubet, Geanakoplos, and Shubik (1989).

Actually, this type of default does not guarantee a particularly high payoff to the strategic defaulting agent, because he gives up his investments. The choice of the settlement mechanism when an agent defaults (keep a fraction of the assets, give up the assets, loose a part of the pledged capitalization) usually relies on the legal context. In our model, we decide that a strategic defaulting agent is giving up his investments, because it is an intermediate way to penalize the default, it does not protect especially the defaulting agents or their victims (the investors). In the introduction, we explained that subprime borrowers were also making strategic defaults because prices of their houses were lower than the amounts of their loans. Long before this crisis, it was known that households could make strategic defaults by filing bankruptcy, as explained by White (1998). The protection of investors and the enforcement law processes influence the choices of the investors and determine

[9]These strategically defaulting agents can be compared to the "black holes" of other models in the network literature, for example Rotemberg (2009).

[10]such as liquidity excess demand or decrease of the value of the assets.

the size of capital markets, as proved by Porta, de Silanes, Shleifer, and Vishny (1997). As a consequence, the choices of the investors (number of investments) are partially depending on this particular mechanism.

As strategically defaulting agents leave the network without reimbursing their creditors, this impacts the solvency of other creditors who may not be able to honor their obligations and may thus become defaulting by contamination. This also reduces the expected payoff of other agents who still remain solvent. Using the contagion mechanism and the structure of the network, we determine the number of defaulting agents D, depending on the number of strategically defaulting agents S.

Strategically defaulting agents are leaving the network, and contaminated defaulting agents collapse, we assume that these two categories of agents are replaced by new agents with the same capitalization C at the beginning of the following period. This allows to keep a constant number of agents over time in the network. This assumption is standard in network models with defaulting agents in a dynamic set-up, see for example in Anand et al. (2012).

Let $G_i^t(A_i^t)$ be the expected gain of agent i from his investments A_i^t at time t, and β the discount factor. The maximization process of each agent can naturally be represented by a Bellman equation. Let $V(A_i^t, L_i^t)$ be the value function of agent i, depending on the number of his assets A_i^t and his number of liabilities L_i^t. At each period, each agent makes a choice, either he stays in the network until next period, or he defaults:

$$V(A_i^t, L_i^t) = \max(\text{strategically default}, \text{stay at least one more period})^{11}, \quad (2.6)$$

$$\boxed{V(A_i^t, L_i^t) = \max\left(L_i^t - A_i^t, \mathbb{E}_t\left[G_i^t(A_i^t) + \beta V(A_i^{t+1}, L_i^{t+1})\right]\right). \quad (2.7)}$$

Proposition 2.3.1. *The value function V is well defined and unique.*

Proof. For simplicity, we suppose that we already know the common number – see Proposition 2.3.2 – m of investments made by agents at each period. We follow

[11] This Bellman equation can be interpreted as that of the McCall (1970) employment search model.

Ljungqvist and Sargent (2004) to solve for the Bellman equation of a McCall type of problem. Let $\mathbb{L}_t$ be the set of incoming links at time t. $\mathbb{L}_t = \mathbb{L}$ is independant of time as soon as the number of investments $A_t = m$ is constant over time. Equation (2.7) can be written as:

$$V(m, L_i^t) = \max\left(L_i^t - m, \mathbb{E}_t\left[G_t(m) + \beta V(m, L_i^{t+1})\right]\right). \tag{2.8}$$

The expectation is linear, which gives:

$$V(m, L_i^t) = \max\left(L_i^t - m, \mathbb{E}_t\left[G_t(m)\right] + \mathbb{E}_t\left[\beta V(m, L_i^{t+1})\right]\right). \tag{2.9}$$

Let T be the operator defined as follows:

$$\begin{aligned} V(m, L_i^t) &= \max\left(L_i^t - m, \mathbb{E}_t\left[G_t(m)\right] + \mathbb{E}_t\left[\beta V(m, L_i^{t+1})\right]\right) \\ &= T(V(m, L_i^t)). \end{aligned} \tag{2.10}$$

Let V_1 and V_2 be two functions, such that for any $0 < L_i^t < N$, $V_1(m, L_i^t) < V_2(m, L_i^t)$. Then,

$$\begin{aligned} T(V_1)(m, L_i^t) &= \max\left(L_i^t - m, \mathbb{E}_t\left[G_t(m)\right] + \mathbb{E}_t\left[\beta V_1(m, L_i^{t+1})\right]\right), \\ &\leq \max\left(L_i^t - m, \mathbb{E}_t\left[G_t(m)\right] + \mathbb{E}_t\left[\beta V_2(m, L_i^{t+1})\right]\right), \\ &\leq T(V_2)(m, L_i^t). \end{aligned} \tag{2.11}$$

This proves that T is monotonic. We check that the operator also satisfies the discounting property. Let α be a positive constant.

$$\begin{aligned}
T(V_1+\alpha)(m,L_i^t) &= \max\left(L_i^t - m, \mathbb{E}_t\left[G_t(m)\right] + \mathbb{E}_t\left[\beta(V_1+\alpha)(m,L_i^{t+1})\right]\right), \\
&= \max\Bigg(L_i^t - m, \mathbb{E}_t\left[G_t(m)\right] \\
&\quad +\mathbb{E}_t\left[\beta V_1(m,L_i^{t+1})\right] + \beta\int \alpha dL_i^{t+1}\Bigg), \\
&= \max\left(L_i^t - m, \mathbb{E}_t\left[G_t(m)\right] + \mathbb{E}_t\left[\beta V_1(m,L_i^{t+1})\right] + \beta\alpha\right), \\
&\leq \max\left(L_i^t - m + \beta\alpha, \mathbb{E}_t\left[G_t(m)\right] + \mathbb{E}_t\left[\beta V_1(m,L_i^{t+1})\right] + \beta\alpha\right), \\
&\leq T(V_1+\alpha)(m,L_i^t) + \beta\alpha. \qquad (2.12)
\end{aligned}$$

By Blackwell's theorem, we know that T is a contraction with modulus β on the complete set of functions on $\mathbb{L}$ with the sup norm. As a consequence, there is a unique fixed point of T. This is the unique value function to the Bellman equation. □

At the end of the period, we distinguish:

- S strategically defaulting agents: $L_i^t - A_i^t > G_i^t + \beta\mathbb{E}_t\left[V(A_i^{t+1}, L_i^{t+1})\right]$;
- $(n-S)$ remaining agents in the network: $L_i^t - A_i^t < G_i^t + \beta\mathbb{E}_t\left[V(A_i^{t+1}, L_i^{t+1})\right]$, among which:
  - H healthy agents, whose situation is such that $L_i^t \leq C + AR_i^t$: the liabilities L_i^t healthy agents pay at the end of the period do not exceed their own capitalization C added to their refunded investments AR_i^t. These agents reimburse their creditors and resist the contamination of the default(s).
  - $D = (n-H-S)$ contaminated defaulting agents: the liabilities contaminated defaulting agents have to repay at the end of the period exceed the refunded investments, added to the capitalization. The counterparties are not paid back.

The estimated number of healthy agents H is therefore a function of the number of agents' investments, the level of capitalization C, the prudential ratio ϕ and the number of strategically defaulting agents S. The following step of the reasoning is to determine the number of investments agents make each period. With these numbers, we can derive the structure of the network and the number of strategic defaulting S and defaulting agents D and the expected payoff of all agents.

2.3.1 Number of investments

The higher the number of strategically defaulting agents, the higher the number of contaminated defaulting agents in the financial network. As a consequence the more strategically defaulting agents, the lower the expected payoff of agents.

Proposition 2.3.2. *There is a unique strategy of investments that minimizes the number of strategically defaulting agents in the financial network: all agents make the same number of investments.*

Proof. The proof can be understood as the optimal choice of the targets of investment. For clarity, we remove the t time index. Agents have the same capitalization C and are limited to $A_{\max} = \left[\frac{C}{\phi}\right]$ investments. Each agent therefore makes between 0 and $A_{\max}$ investments. Because no reciprocal investment is possible, this leads to $A_{\max} \leq \frac{n}{2}$. Let $\bar{A}$ be the average number of investments made by an agent in the network. A_i is the number of investments made by agent i. We order the agents owing to their number of investments. Let agent 0 be the "lowest" investor, i.e. the one making the lowest number of investments. On the opposite, agent n is the "largest" investor.

- Suppose that agent 0 is making exactly 0 investment. His expected payoff from investing is thus 0. If he receives a least one investment, strategically defaulting guarantees a strictly positive payoff. As a consequence, no creditor will be paid back at the end of the period. Obviously, no one will make an investment towards this agent. Agent making 0 investment is not a good target for investment, the agent is said to be not "attractive". Similarly, if agent 0 were to be a low investor: $0 < A_0 << \bar{A}$, any other agent of the network would fear this agent as a potential strategical defaulter. As a consequence, he will not receive any investment either.

- We can suppose $0 < \bar{A} \leq A_n \leq A_{\max}$. If $A_n = \bar{A}$, all agents make the same number of investments. Any other agent of the network may consider to invest towards agent n because he expects agent n not to be – *a priori* – a strategically defaulting agent. If all agents of the network act this way, agent n will receive approximately $(n-1)$ investments, and he is likely to become a strategically defaulting agent because $A_{\max} << n-1$. Thus investing towards this agent is not a good strategy.

Both the lowest and the largest investors are not attractive. Investors restrict their choices to the $(n-2)$ remaining agents of the network. We consider the new set of agents, from agent 1 to agent $n-1$, and we apply the same reasoning, until the remaining agents are making exactly the same number of investments.

To decrease the overall number of strategically defaulting agents, agents are willing to distribute evenly the investments among themselves. Any agent which is not receiving investments because he is not attractive (too many or too few investments) forces the other agents to make their investments towards other more attractive agents. This situation creates some attractive agents receiving too many investments: $L_i >> \bar{A}$ and possibly makes them strategically default. To minimize the number of strategically defaulting agents, the best way is to maximize the number of attractive agents.

To conclude, all agents have to make the same number of investments. From there A_t represents the common number of investments by agents at time t. □

2.3.2 Number of strategically defaulting agents

We showed that all agents make the same number of investments. We can determine precisely the number of strategically defaulting, contaminated defaulting and healthy agents in the economy as a function of the number of investments by agent.

Agent i strategically defaults if and only if $L_i^t - A_i^t > G_i^t + \beta\mathbb{E}_t\left[V(A_i^{t+1}, L_i^{t+1})\right]$. When at least one agent is strategically defaulting in the economy, there might exist contaminated defaulting agents because some investments will not be paid back. Logically, the outcome depends on the level of capitalization of agents. The expected payoff of an agent depends on his total number of investments, and on the financial situation of the counterparties of his investments. Let $\mathbb{P}(H)$ be the probability to reach a healthy, $\mathbb{P}(D)$ a contaminated defaulting, or $\mathbb{P}(S)$ a strategically defaulting agent with an investment. We consider agent i. The expected payoff $G_i(A_i)$ from investments of agent i (risk neutral), who does not strategically default, is derived in the following proposition[12]:

Proposition 2.3.3. *When all agents make each A investments, the expected payoff of an agent staying in the network is:*

$$\boxed{G(A) = A\left((e+1)\mathbb{P}(H) - 1 + \mathbb{P}(S)\right). \qquad (2.13)}$$

Proof.

$$G(A_i) = \mathbb{E}\left[\sum_{j\in H} e\mathbb{1}_{i\to j} - \sum_{j\in(D\cup S)} 1\mathbb{1}_{i\to j} + \sum_{j\in S} 1\mathbb{1}_{j\to i}\right]. \qquad (2.14)$$

The last term of the previous equation $\mathbb{P}\left((j\to i)\cap(j\in S)\right)$ corresponds to the probability that agent j is a strategically defaulting agent, and is investing towards agent i. Before the end of the period, he leaves the network and gives up his own investment to agent i.

$$G(A_i) = A_i\left[e\mathbb{P}(H) - \mathbb{P}(D) - \mathbb{P}(S)\right) + \sum_{j\neq i}\mathbb{P}\left((j\to i)\cap(j\in S)\right]. \qquad (2.15)$$

By definition, $\mathbb{P}(D) = 1 - \mathbb{P}(H) - \mathbb{P}(S)$, there are $(n-1)$ directions for each investment, this leads to $\mathbb{P}(j\to i) = \frac{A_j}{n-1} = \frac{A_i}{n-1} = \frac{A}{n}$, because all agents are making exactly A investments. □

If no agent is strategically defaulting, there is no financial distress, and for each agent $A_i{=}AR_i$. In this case, the payoff for all agents is $G_{\max} = Ae$. In this

[12] To simplify notations, we removed the t time index.

particular context, no agent is strategically defaulting except if at time T:

$$L_i^T - A > \sum_{t \geq T} \beta^{t-T} G_{\max} = \frac{Ae}{1-\beta}. \tag{2.16}$$

This condition is equivalent to: $L_i^T > A(1 + \frac{e}{1-\beta})$. Using this condition we determine the number of initially strategically defaulting agents. If some agents are strategically defaulting because condition (2.16) is fulfilled for some i, we call them initially strategically defaulting agents, because they will not pay back their investments, and they will create contaminated defaulting agents. For this reason, by equation (2.13), we know that $G(A) < G_{\max}$. Because $G(A)$ is lower than $G_{\max}$, it might happen that some other agents strategically default. To calculate the exact number of strategically defaulting agents, we must evaluate the term $V^* = \mathbb{E}_t \left[V(A_i^{t+1}, L_i^{t+1}) \right]$. By induction, we have:

$$\begin{aligned} \mathbb{E}_t \left[V(A, L_i^{t+1}) \right] &= \mathbb{E}_t \left[\max \left(L_i^{t+1} - A, G(A) + \beta \mathbb{E}_{t+1} \left[V(A, L_i^{t+2}) \right] \right) \right] \\ &= \mathbb{E} \left[\max \left(L_i - A, G(A) + \beta V^* \right) \right]. \end{aligned} \tag{2.17}$$

In the previous equation, we remove the t subscripts, because there is no alea.

Proposition 2.3.4. *The term $V^* = \mathbb{E}[V(A, L_i^{t+1})]$ represents a reservation payoff of staying in the network, and postponing the decision to default strategically to the next period. V^* is well defined, and only depends on A.*

This proposition is crucial. The value of V^* influences the number of strategically defaulting agents, and therefore the number of contaminated defaulting agents, and the expected payoff of the agents. For this reason V^* "influences its own value". The following proof shows that V^* exists and is unique. Otherwise, we would not have a solution to the Bellman equation (2.7).

Proof. We know that $V^* \geq 0$, (0 is the case of no investment or risk loving agents) and $V^* \leq G_{\max} \sum_{t \geq 0} \beta^t$. Let us write $V^* = \mathbb{E} \left[V(A, L_i^{t+1}) \right]$. The previous equation

(2.17) becomes:

$$
\begin{aligned}
V^* &= \mathbb{E}\left[\max\left(L_i^{t+1} - A, G(A) + \beta V^*\right)\right], && (2.18)\\
&= \mathbb{E}\left[\left(L_i^{t+1} - A\right)\mathbb{1}_{L_i^{t+1}-A-G(A)\geq\beta V^*}\right] + \mathbb{E}\left[\left(G(A) + \beta V^*\right)\mathbb{1}_{L_i^{t+1}-A-G(A)<\beta V^*}\right]\\
. &= \sum_{k=A+G(A)+\beta V^*}^{n-1} k\mathbb{P}(L_i^{t+1} = k) + \sum_{k<A+G(A)+\beta V^*} \left(G(A) + \beta V^*\right)\mathbb{P}(L_i^{t+1} = k),\\
&= \sum_{k=A+G(A)+\beta V^*}^{n-1} k\mathbb{P}(L_i^{t+1} = k) + \left(G(A) + \beta V^*\right)\sum_{k<A+G(A)+\beta V^*} \mathbb{P}(L_i^{t+1} = k).
\end{aligned}
$$

No agent would invest if the expected payoff was not positive. The value function is also positive. This explains the second equality.

Agents such that $L_i^{t+1} - A > G(A) + \beta V^*$ are exactly the strategically defaulting agents, while others remain in the economy. There are exactly $n - S^*$ remaining agents in the network, where S^* is the equilibrium value of the number of strategically defaulting agents. We can deduce that

$$
\sum_{k<p+G(p)+\beta V^*} \mathbb{P}(L_i^{t+1} = k) = \frac{n - S^*}{n}, \tag{2.19}
$$

and

$$
\sum_{k\geq p+G(p)+\beta V^*} \mathbb{P}(L_i^{t+1} = k) = \frac{S^*}{n}. \tag{2.20}
$$

Since all the terms $\mathbb{P}(L_i^{t+1} = k)$ can be calculated, at least numerically, we can also calculate the values of the following function, which represents the cumulative distribution function of the liabilities:

$$
L(x) = \sum_{k=x}^{n-1} \mathbb{P}(L_i^{t+1} = k). \tag{2.21}
$$

From the last function, we deduce x^* such that $L(x^*) = \frac{S^*}{n}$. This gives the value of $x^* = A + G(A) + \beta V^*$. We can determine $G^*(A) = G(A)$ as a function of S^*, because using S^*, we can calculate D^* and deduce G^*. The global equation

(2.18) on V^* gives us the theoretical relation between S^* and V^*:

$$V^* = \left(\sum_{k=L^{-1}(\frac{S^*}{n})}^{n-1} k\mathbb{P}(L_i^{t+1} = k) \right) + \frac{n - S^*}{n} \left(G^*(A) + \beta V^* \right). \qquad (2.22)$$

For each value of S^*, we can calculate a corresponding value of $V^* = V^*(S^*)$. We derive the term $G^*(A) + \beta V^*$. Given the distribution of L_i^t, we can determine the expectation $\overline{S}$ of the number of agents such that $L_i^t - A > G^*(A) + \beta V^*$. When $S^* = \overline{S}$ the solution is obtained. This proves that there exists a well-defined solution. Given A, there is a unique $(S^*, D^*, H^*, G^*(A))$. $\square$

In practise, we construct a sequence, converging to the values of S^*, $G(S^*)$ and V^*: we start from S_1 the initially strategically defaulting agents of the network, those such that relation (2.16) is verified. Then we calculate over the network the spread of financial distress, i.e. the number of contaminated defaulting D_1 and healthy H_1 agents. This gives G_1 by equation (2.13). Using (2.22), we calculate V_1. Using V_1 and the Bellman equation (2.7), we determine the new number S_2. Then we calculate, D_2 and H_2, and G_2. Using (2.22) we calculate S_3, and so on. This sequence of (S, D, H, G) converges to the solution.

We proved that agents make the same number of investments at each period. We know how to calculate the number of strategically defaulting, contaminated defaulting and healthy agents, as well as the expected payoff function of the agents, depending on the number of investments by agent. Agents optimize their expected payoff by choosing the number of investments they make. The regulator limits the number of investments by the prudential ratio to maximize the number of healthy agents. In the model, because investments' amounts are normalized to 1, the number of investments determines the connectivity of the network. Increasing the capitalization ratio decreases the connectivity of the network. ϕ does not *a priori* limit the amount of investments, however, there exists an internal threshold within any financial intermediary, which limits the financial position of the intermediary on any financial contract with one counterparty. As a consequence, ϕ does not only limit the number but also the volume of the transactions of agents in the network.

In the sequel, we study how the agents' problem (optimize the discouted sum of profits), and the regulator's problem (maximization of the number of healthy agents) are compatible. To simplify the problem, we analyze first the short term case, where agents play only one period. Then we see how the infinite periods game changes the results. Intermediary cases, when agents are myopic and base their expectations on a limited number of periods, will be presented briefly in the last section.

2.4 Simulation of the one-period network: short-term gains

A description of the method used to produce simulations is provided in Appendix B of the chapter.

2.4.1 Initially strategically defaulting agents

Technically, the one-period network is equivalent to consider $V^* = 0$ in the Bellman equation (2.7). We remove the t subscripts and the recursive equation becomes:

$$\boxed{V(A, L_i) = \max(L_i - A, G(A)). \qquad (2.23)}$$

The payoff equation remains the same:

$$G(A) = A\left((e+1)\mathbb{P}(H) - 1 + \mathbb{P}(S)\right). \tag{2.24}$$

The equation determining the number of initially strategically defaulting agents slightly changes and becomes:

$$L_i - A > G_{\max} = Ae. \tag{2.25}$$

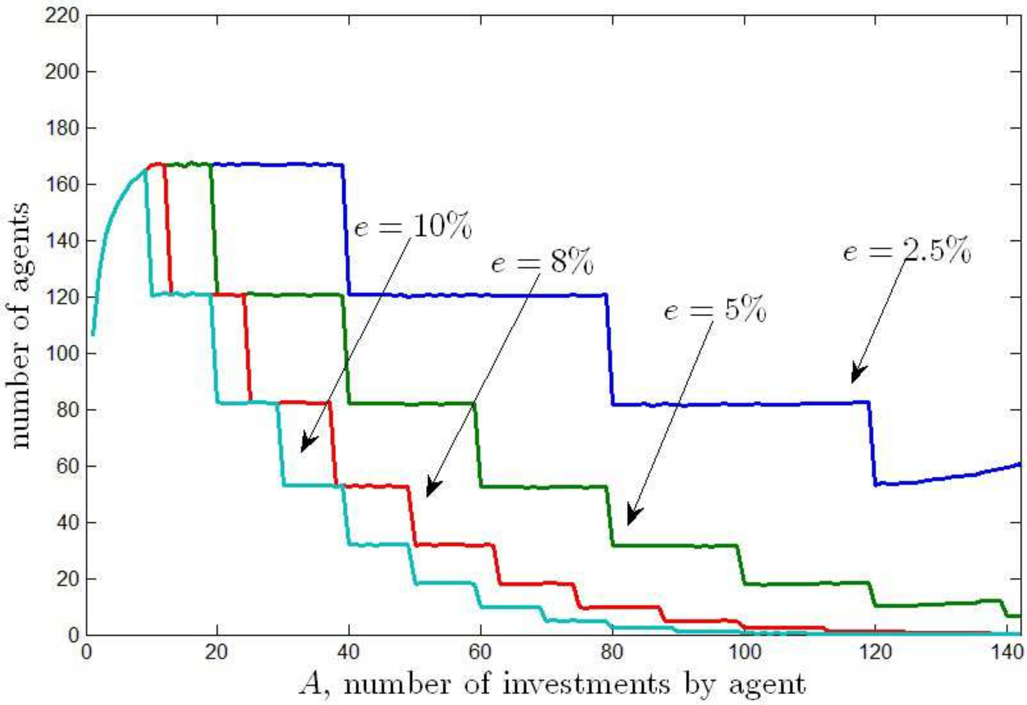

Figure 2.2: Number of initially strategically defaulting agents w.r.t. the number of investments by agent.

Figure 2.2 is plotted using the following parameter values: $n = 400$, $\phi = 7\%$, and $C = 10$ (arbitrary units). The maximal number of investments by agent is determined by the value of the prudential ratio: $A_{\max} = \frac{C}{\phi} = 142$. We simulate over a large number of networks (≈ 2000) the average number of initially strategically defaulting agents of the network depending on the payoff by investment from $e = 2.5\%$ to $e = 10\%$ on Figure 2.2.

All the curves representing the number of initially strategically defaulting agents have the same shape:

- After an initial increase, each curve is decreasing with respect to the number of investments by agents,
- the curves are not smooth: there are regular thresholds of decreases.

These observations can be easily explained. For each value of e, the regularity of the "sudden decreases" of the curve corresponds to values of A such that Ae is an integer. Indeed, the number of liabilities minus the number of assets is obviously an integer, therefore the comparison threshold between the net gain from strategically defaulting and the gain from investments does not change when Ae keeps the same integer part. The periodicity in terms of investments is calculated the following way: $\pi_e = \frac{1}{e}$. For the first C investments, there is no constraint about the allocation of investments because each agent satisfies $A_i \leq C$: the allocation is purely random. When the number of investments by agent increases and remains smaller than C, the probability to get more investments than the average increases: the number of strategically defaulting agents increases – as long as Ae keeps the same integer part. Once the average number of investments by agent is larger than C, each agent making A investments must receive at least $(A - C)$ investments in order to have the balance sheet condition (2.5) satisfied. This situation is equivalent to distribute randomly C investments made by agent over the whole network and $(A - C)$ investments exactly received and made by each agent[13]. The probability to get x more investments than A is therefore constant with respect to A, whatever the value of $A > C$. The global decrease of the curve is due to the increasing gains due to investments: Ae.

2.4.2 Viability of the short term network

Using the number of initially strategically defaulting agents, we calculate the number of defaults in the network. These defaults reduce the expected gains of the other agents, who therefore have an incentive to strategically default as well. We determine the stationary equilibrium number of all strategically defaulting agents in the network. For Figure 2.3, we adopt $e = 5\%$. With the exact number of strategically defaulting agents, we also know the number of contaminated defaulting agents and the number of healthy agents in the network.

[13] In terms of graph theory, we can say that the in-degree of each node is larger than $(A - C)$ and the out degree is exactly A.

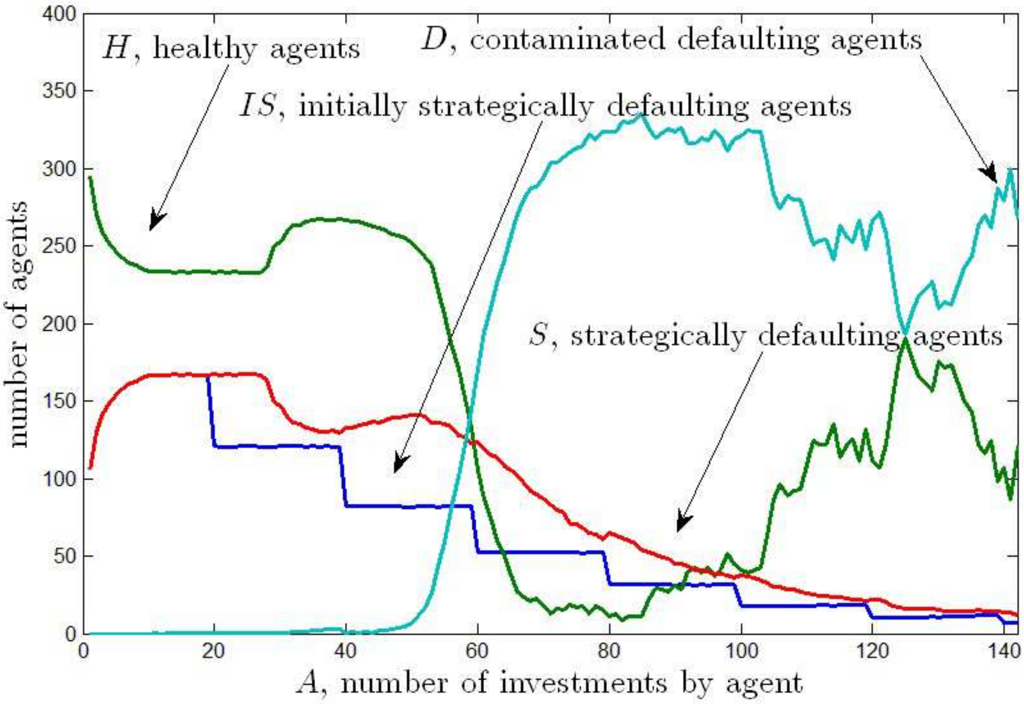

Figure 2.3: Number of healthy and defaulting agents w.r.t. the number of investments by agent

To determine the exact number of investments agents make, we plot the expected payoff by agent depending on the number of investments on Figure 2.4, for different values of the prameter e.

On Figure 2.4, we observe the expected payoff of agents as a function of the number of investments by agent. For $e = 10\%$, the expected payoff is strictly increasing with respect to the number of investments by agents. As a consequence, the unique equilibrium choice of agents is to make exactly $A_{\max} = \frac{C}{\phi}$ investments.

If we consider the curves for which $e = 2.5\%$ or $e = 5\%$, there exists a zone such that the expected payoff is lower than or equal to zero. Agents do not enter the network if the expected payoff is negative. For 40 investments by agents, the expected payoff is strictly positive and reaches its maximal value. Agents will therefore coordinate and make exactly 40 investments each. This situation illustrates some features of the recent crisis. Agents make investments, but it is

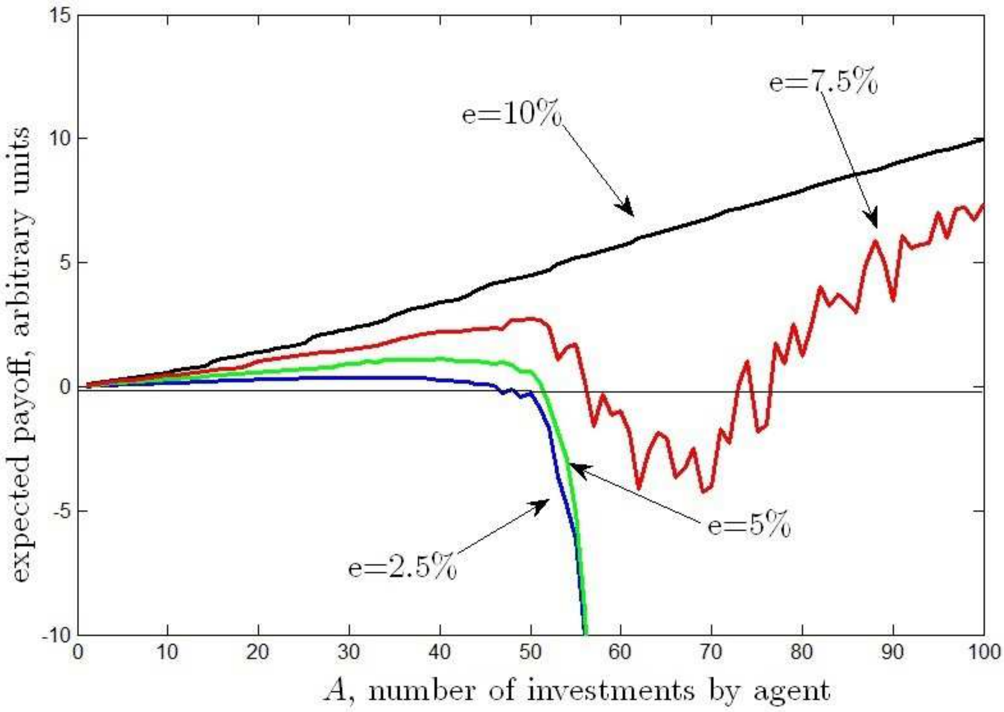

Figure 2.4: Expected payoffs w.r.t. the number of investments by agent

optimal not to invest as much as possible, because it generates too big distresses, and negative payoffs. During the interbank crisis in 2008, banks refused to lend (invest in the model), fearing that their counterparties may be in financial distress, which generated a liquidity squeeze.

For the intermediary value $e = 7.5\%$ there is a "tricky" situation, the expected payoff is positive for $A \leq 55$ to $A \geq 75$ investments by agents. The choice of the agents in terms of expected payoff depends on the value of the prudential ratio. If $A_{\max} \leq 85$ which is equivalent to $\phi \leq 12\%$, the agents choose to make $A_{\max}$ investments. On the contrary if $\phi > 12\%$ agents choose to make $A = 55$ investments.

We state the conclusions of the one-period networks:

- for high values of the return on investments ($e > 8\%$), agents make the highest number of investments, limited by the prudential ratio ϕ;
- for low values of the return on investments, agents choose to make a lower number of investments than the theoretical limit fixed by the regulator. This may represent liquidity problems;
- for very low values of the return on investment, risk-neutral agents do not play in the network. Risk-seeking agents may invest even if the expected payoff is negative because there is still a positive probability to be one strategic defaulting agent, whose net gain is strictly positive;
- agents enter the network and make investments even if there is a net positive number of strategically defaulting or contaminated defaulting agents.

From a social welfare perspective, the regulator seeks to minimize the number of strategically defaulting and contaminated defaulting agents, and maximize the number of healthy agents. On the previous example, using Figure 2.3, we remark that this objective may correspond to the maximization of the expected payoff of agents. However, these two objectives may be incompatible.

- For some parameter values, the maximization of the expected payoff and the maximization of the number of healthy agents are conflicting, see for example on Figure 2.5, $\phi = 10\%$ and $e = 8\%$: The expected payoff is on average increasing with the number of investments by agent while the number of contaminated defaulting agents reaches its maximum for some bounded value. Agents want to make $A_{\max} = \frac{C}{\phi}$ investments while the regulator would prefer the agents to make either more than 80 or less than 50 investments. The only solution for the regulator is to choose either $\phi < 12.5\%$ or $\phi > 20\%$.
- To achieve his goals, the regulator is interested in increasing the value of the prudential ratio ϕ to limit the number of investments by agent, but this is likely to reduce the expected gains of agents.

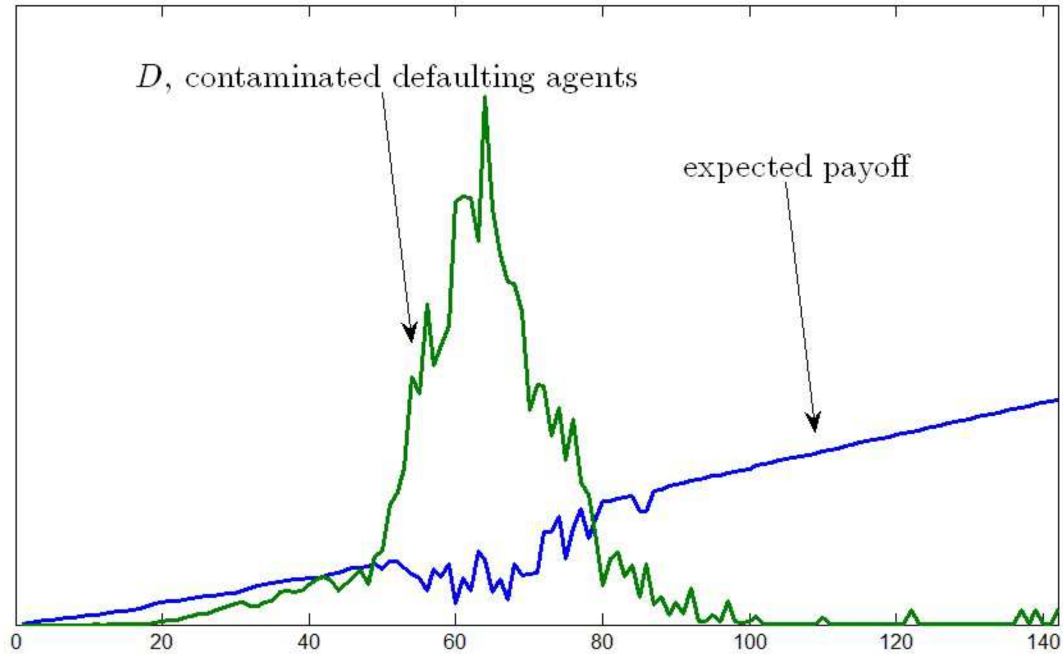

Figure 2.5: Optimization of the expected payoff vs. minimization of the number of defaulting agents

- What is worse between strategically defaulting and contaminated defaulting agents? The strategically defaulting agents make the choice to default, initiate the defaults, and are responsible for the problems, and the contaminated defaulting agents are consequences of the strategically defaulting agents. On the one hand the regulator tries to banish strategically defaulting agents even if there are few contaminated defaulting agents, on the other hand the regulator tries to limit the proliferation of financial distresses coming from a few number of strategically defaulting agents: the regulator arbitrates (trade-off). Contaminated defaulting agents are more costly than strategically defaulting agents, because they do not refund any of their creditors, while strategically defaulting agents give up their own investments.

2.5 Infinite time model

The one-period network yields a very large number of defaulting agents, which is quite unrealistic. We therefore concentrate on the more realistic case of an infinite time network. To determine the number of strategically defaulting agents, we introduce the $V^* \neq 0$ parameter in the Bellman equation (2.7). Even if there is a well-defined expression for V^*, all the probabilities and also the contamination are difficult to evaluate. V^* is estimated by a recursive method.

2.5.1 Determination of the postponing threshold V^*

As we already know, from time T, $V^* > \sum_{t>T} \beta^{t-1} G_i^t(A^t)$. Starting from $V^* = 0$ we deduce A^* such that $G^t(A^*)$ is maximal. Then we replace V^* by $\frac{1}{1-\beta} G^t(A^*)$ and we compute again the new expected numbers of strategically defaulting, contaminated defaulting and healthy agents, and we also deduce the expected gain function $G^t(A^t)$. Then we determine the new A^* which maximizes G^t. We replace V^* by $\frac{1}{1-\beta} G^t(A^*)$, until there is convergence. We assume that this lower bound estimation of V^* is close to its real value as long as the probability $\mathbb{P}(S)$ to reach a strategically defaulting agent with an investment remains very low.

To get the best accurate value of V^*, there is an important discussion about parameters e and β. The discount factor β is not supposed to change over the period and is common to all financial entities. It is related to the long-term risk free rate r_∞. On the opposite, e is the return on a one-period repaid investment. It is related to the short-term interest rate.

Suppose we know $G_{\max}$. Then $\sum_{t>0} \beta^t G_{\max} = \frac{\beta}{1-\beta} G_{\max}$. If the rate of return of the capital is constant, then $\beta = \frac{1}{1+e}$. If also $G_{\max}$ is proportional to e: $G_{\max} = Ae$, the network is indifferent to the value of e. For high values of e, when almost no one is strategically defaulting, this is true. On the opposite case, for low values of the return on investment, $G_{\max} \neq Ae$, because there are strategically defaulting and contaminated defaulting agents. Introducing a net shift V^* in the maximization equation will modify the behavior of agents: there will be less strategically defaulting agents.

We assume that the long term interest rate is $r_\infty = 2.5\%$[14]. In this case $\frac{\beta}{1-\beta} = 49$. To evaluate V^*, we shall consider the corresponding gain. For $e = 2.5\%$ the maximal expected payoff of the short-term network is $G \approx 0.38$. A lower bound for V^* is $V^* = 18$. As shown on Figure 2.7, taking $V^* \geq 10$ is large enough to completely eliminate strategically defaulting agents. As a consequence, there are no contaminated defaulting agents, all financial agents are healthy.

We show the results of simulations, obtained for different values of V^*, from the one period model $V^* = 0$ to $V^* = 10$. This represents the state of the network depending on the weight financial agents put on the future payoffs.

As we observe on Figure 2.6, as soon as $V^* > 5$, the number of initially strategically defaulting agents is highly reduced compared to the one-period case. When $V^* > 10$, there are no initially strategically defaulting agents anymore.

On Figure 2.7, we present the total number of strategically defaulting agents; S remains very close to the number of initially strategically defaulting agents. This is easy to understand: since $V^* >> G_{\max}$, the effect of the initially strategically defaulting agents on $G_{\max}$ is negligible compared to the value of V^*. Then we derive the number of healthy agents.

[14]We choose to adopt realistic parameters values like Nier et al. (2007), even if some other range of parameters may lead to more impressive results. For the number of banks, we can take into account the work of Bech and Atalay (2010). The same remark –adopting realistic values– comes at the end of the fourth chapter, and has been a motivation for the last chapter. As a matter of fact, this remark will never percolate further than the current dissertation, because concerned articles are currently published. For example, in the model of Anand et al. (2012), there could be a debate on the payoff of the global game in the case where an agent rollovers a contract. If the bank remains solvent, the creditor earns $(1 - a)$ and if the bank defaults the creditor earns a. We could expect that what a creditor earns $(1 - a)$ is related to the interest rate $(+/-$ risk premium) when the counterparty is solvent, while in case of default, a creditor would loose a large proportion of his investment $> \frac{1}{2}$. This would give the range of a, from 0.5 to 1 but rather close to 1. Unfortunately, for $a > 0.7$ the network does not exist, agents choose to foreclose, and there are almost no connections at the stationary state. In the following of the dissertation, we keep realistic parameters' values, sometimes to the detriment of more impressive performances.

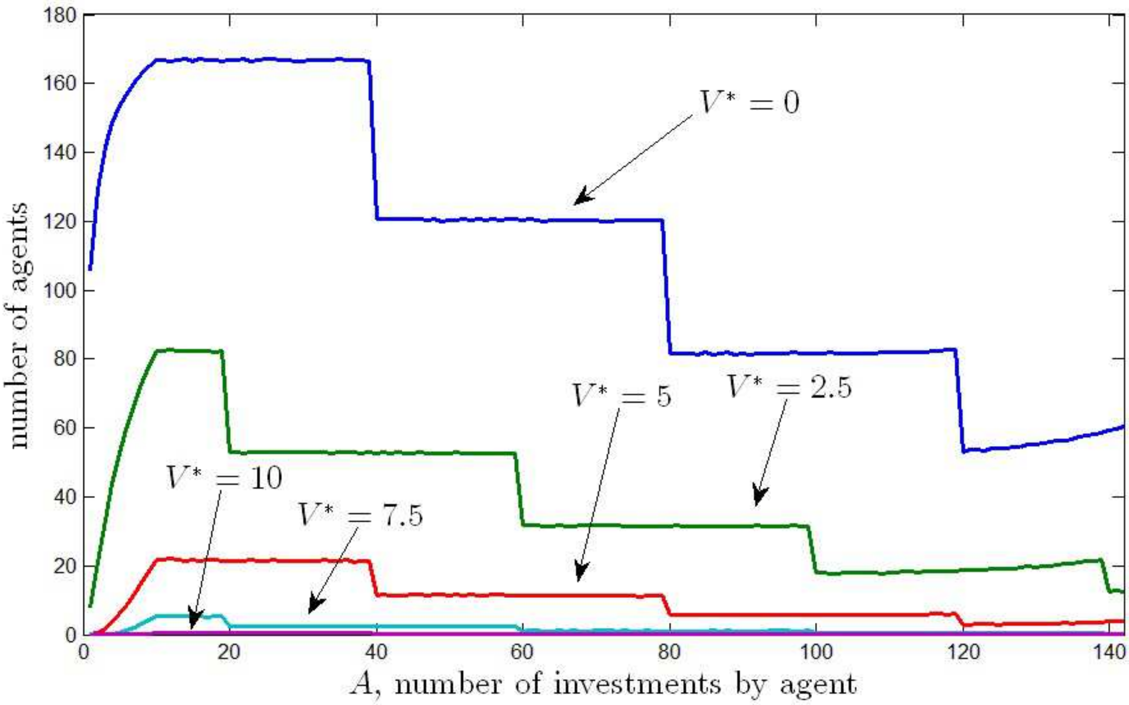

Figure 2.6: Number of initially strategically defaulting agents w.r.t. the number of investments

2.5.2 Global results

In this section, all the simulations correspond to $e = 2.5\%$. On Figure 2.8, the number of healthy agents is quite high, as soon as $V^* > 2.5$. To find the number of investments by agents, we show the estimated expected payoff by agent on Figure 2.9.

Given the simulations on the expected payoffs depending on the value of V^*, we remark that the payoff is strictly increasing if $V^* > 5$. This proves that when $n = 400$ and $C = 10$ in the infinite time horizon, the financial network reaches by itself a stable and healthy situation. For $V^* = 5$ the expected payoff is on averge increasing, but decreases when the number of investments by agents exceed 120. This curious behavior is the representation of the systemic risk. To understand this key issue, we zoom in Figure 2.10 on the number of healthy agents for $V^* = 5$.

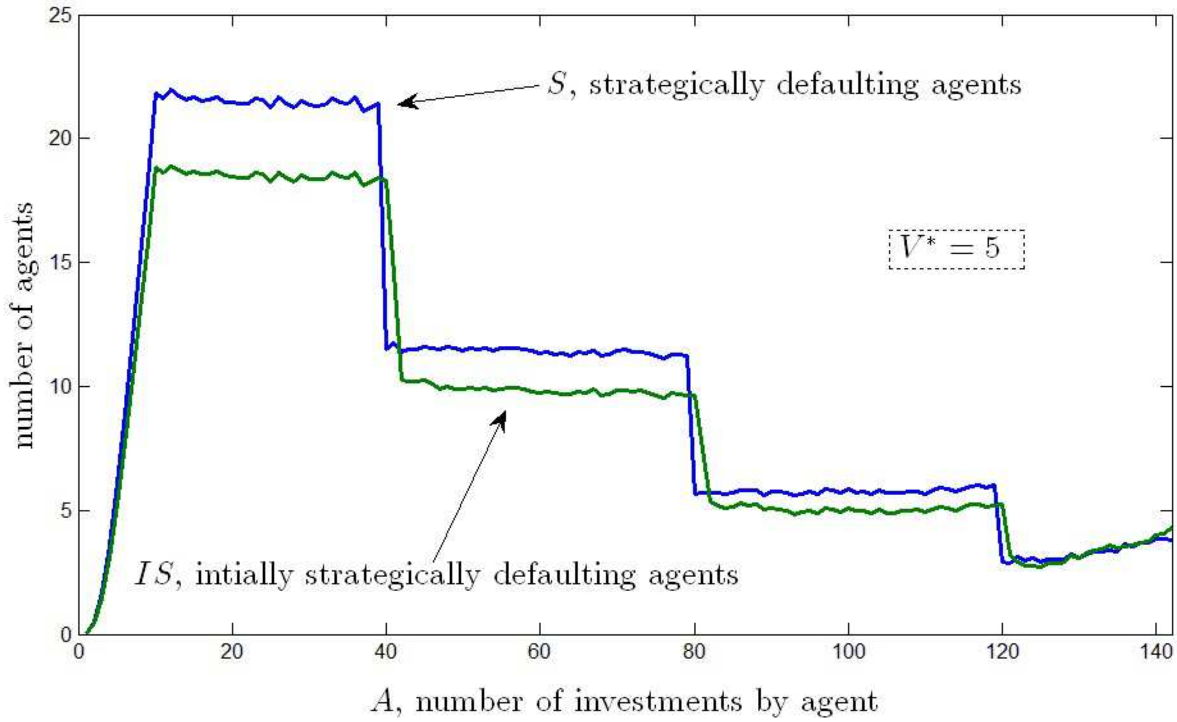

Figure 2.7: Number of strategically defaulting agents w.r.t. the number of investments by agent

The curve of healthy agents is obtained by taking the average number of healthy agents over a large number of simulations. The curve is quite smooth for $A < 100$. Indeed, the difference between the total number of agents $n = 400$ and the number of healthy agents corresponds to the number of strategically defaulting agents, as drawn on Figure 2.7. However for $A > 100$ there are a few strategic defaulting agents: $S < 4$. As a consequence, there is a net positive number of contaminated defaulting agents. Precisely, among the different simulations, a large number of them have a high number of healthy agents: $H > 390$ and some others have 0 healthy agents, and $D > 390$ contaminated defaulting agents. Taking the average over these simulations makes the average curve very noisy, even with ≈ 1500 simulations. This proves that systemic risk exists, because a few number of simulations lead to the whole default of the economy. Using the simulations, we can deduce the frequency of systemic defaults: in the range 100 to 115 investments

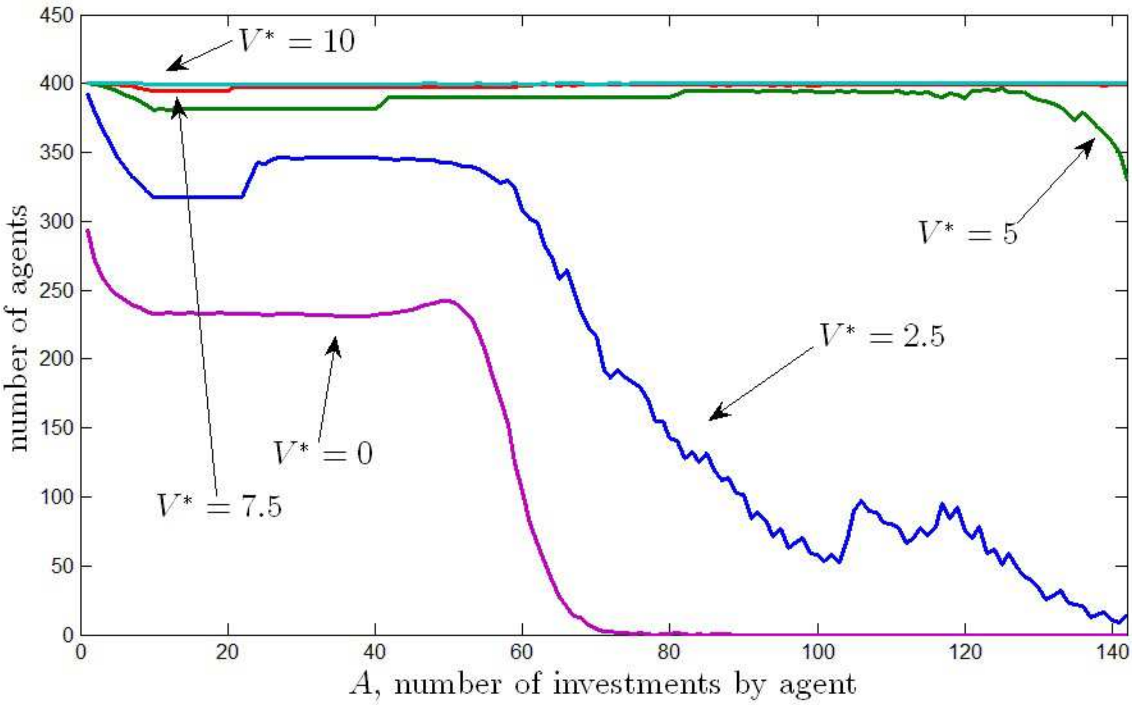

Figure 2.8: Number of healthy agents w.r.t. the number of investments by agent

by agent, there is on average 1 situation of systemic default on 1000 simulations, which represents a risk of 0.1%. From 115 to 130 investments by agent, there is a systemic risk of 0.2%. Above 130 investments by agent, there is a systemic risk of 0.5%. In this case, fixing the prudential ratio to $\phi = 10\%$ avoids to exceed $A = 100$ investments by agents, and prevents sytemic risk, without reducing by a large amount the expected payoff of agents. The prudential ratio eliminates the systemic risk.

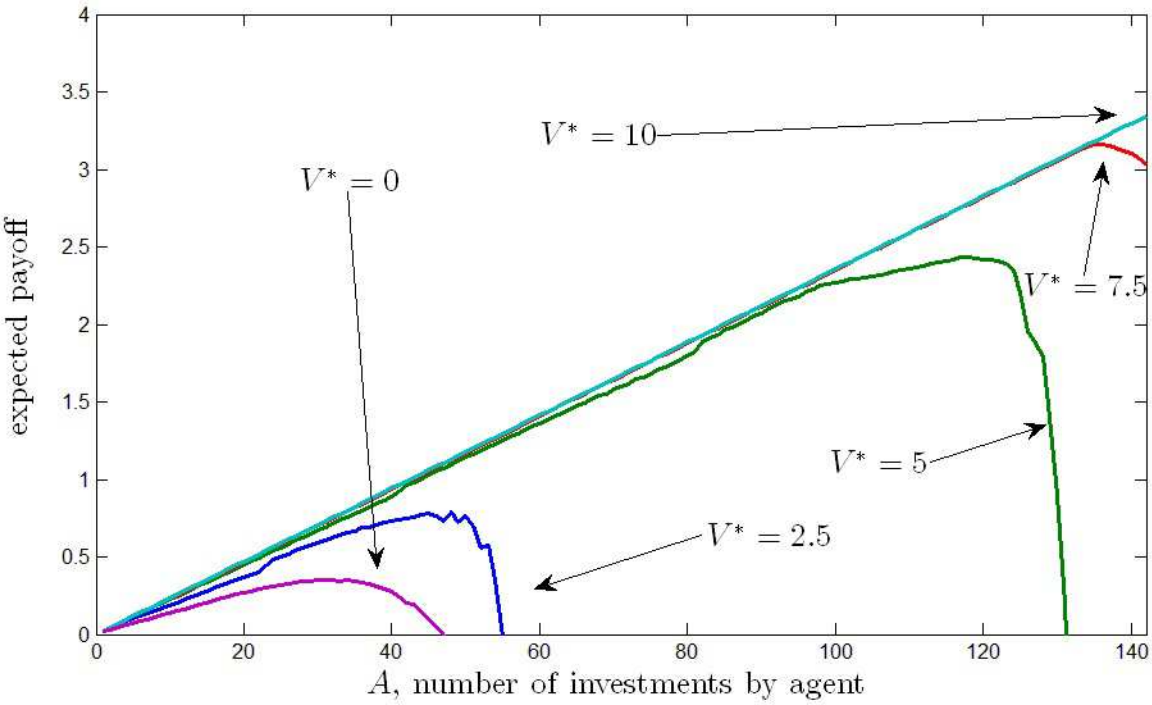

Figure 2.9: Expected payoff (estimated) w.r.t. the number of investments by agent

- Long-run networks have a lot less strategically defaulting agents than short-term networks.
- When agents are risk neutral and optimize over their infinite lifetime, when the discount factor and the return on investment can be linked: $\beta = \frac{1}{1+e}$, the network reaches a state containing only healthy agents.
- The prudential ratio ϕ limits the number of investments by agents. If agents adopt a myopic behavior, some cases of default appear for a high level of investments by agent, this represent the systemic risk. Unlike conclusion of most other networks models, more complete networks have a higher systemic risk than less dense networks. Setting properly the value of the prudential ratio ϕ avoids systemic risk.

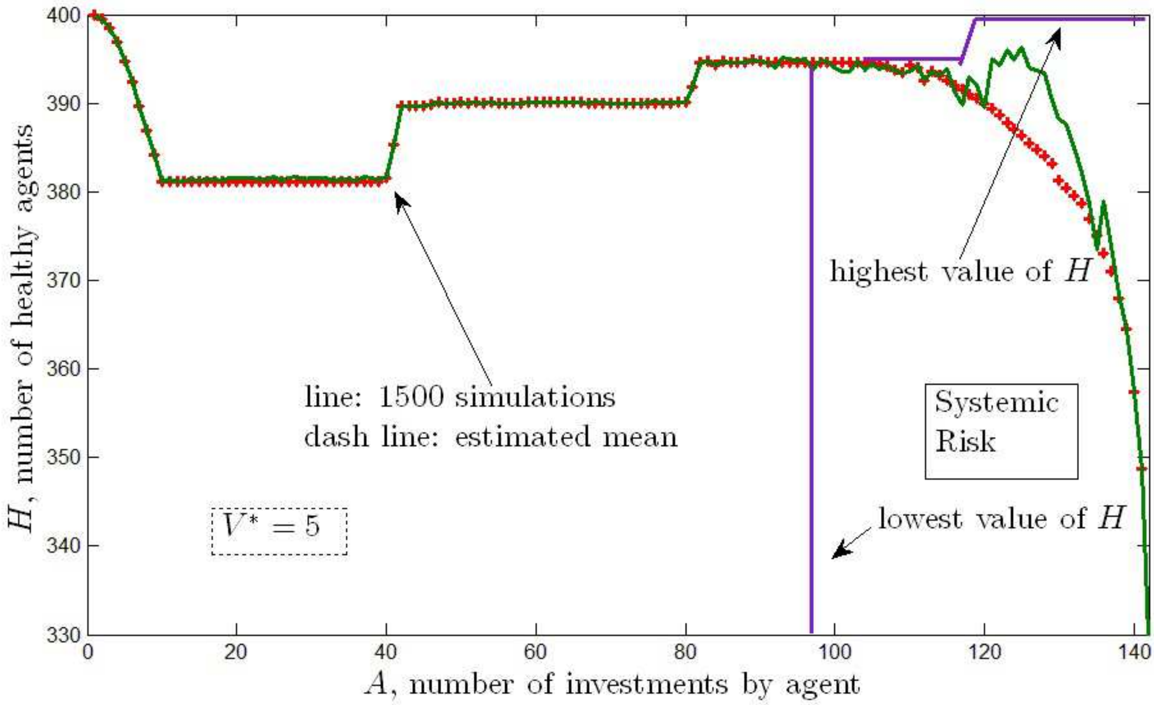

Figure 2.10: Healthy agents for $V^* = 5$

If agents attach less importance to their future payoffs, we say that they adopt a myopic behavior. This myopic behavior may represent agents who expect to achieve their payoffs in finite time, or agents who fear the long-term equilibirum of the network (risk of crisis). This is equivalent to reduce V^*. Precisely, we define a myopic agent as an agent who only takes into account the gains of the current period and the 3 following ones. We determine in the next section the performance of the prudential ratio on the different situations: short-term, myopic, long-run.

2.5.3 Robustness of the ratio

In the two previous sections, we studied the case of a financial network containing $n = 400$ agents, all of them owning a capital of $C = 10$ units. We raise the question whether the results and the policy implication change with respect to these variables. By simulation, we prove that the results hold when n changes, as soon as n remains large enough. When the capitalization of agents changes, the results evolve. When agents have a lower capitalization, the number of assets is close to the number of liabilities: agents make A investments and must have at least $A - C$ liabilities. As a consequence we could expect less strategically defaulting agents. However, with less capital and the same prudential ratio, the maximum number of investments is lower. Table 2.1 determines which effect dominates. We fix $e = 2.5\%$ and $\beta = 0.98$, we let C vary.

On Table 2.1, we compare the results obtained with different values of the capitalization, for two choices of the prudential ratio 5% and 10%, and for three types of behaviors of agents: one-period optimization, infinite lifetime and myopic behavior. In the table, we show the results of simulations for different capitalizations: from $C = 4$ to $C = 10$ (arbitrary units). We determine the maximal number of assets $A_{\max}$, which only depends on the capitalization C and the prudential ratio ϕ: $A_{\max} = \frac{C}{\phi}$. We deduce the theoretical maximal gain $A_{\max}e$. Then, by induction, we determine the three terms V^*, G^* and A^*, which represent the optimal choices of the agents.

To determine whether ϕ is effective, we distinguish three cases:

- $A^* = A_{\max}$, ϕ is fully effective because agents make the maximum number of investments. In this case, the prudential ratio prevents to reach a too connected state, which could present a systemic risk, especially when agents are myopic.
- $A^* << A_{\max}$, ϕ is useless, agents make a small number of investments. There are liquidity problems. Agents could make more investments, but they do not make them, because they are afraid of contamination. The network would be in bad state (systemic failure) for a high number of investments by agents. This case would be emphasized by risk-averse agents.
- $A^* \approx_< A_{\max}$, ϕ should be a little reduced to avoid risk of defaults.

Let us focus on a particular case: for $n = 400$, $C = 5$ and myopic behavior, corresponding to the two red lines of Table 2.1. When $\phi = 5\%$, agents restrict

C	period	$\phi\%$	$A_{\max}$	$A_{\max}e$	G^*	V^*	A^*
10	long	5	200	5	5	200	200
10	myopic	5	200	5	5	15	200
10	short	5	200	5	0.75	00	35
10	long	10	100	2.5	2.5	122.5	100
10	myopic	10	100	2.5	2.33	7	100
10	short	10	100	2.5	0.75	00	35
7	long	5	140	3.5	3.5	171	140
7	myopic	5	140	3.5	3.3	10	130
7	short	5	140	3.5	0.21	0	19
7	long	10	70	1.75	1.75	86	70
7	myopic	10	70	1.75	1.65	5	66
7	short	10	70	1.75	0.21	0	19
5	long	5	100	2.5	3.5	171.5	100
5	myopic	5	100	2.5	2.1	6.3	85
5	short	5	100	2.5	0.12	0	9
5	long	10	50	1.25	1	49	50
5	myopic	10	50	1.25	0.14	0.4	9
5	short	10	50	1.25	0.12	0	9
4	long	5	80	2	2	98	80
4	myopic	5	80	2	0.08	0.25	6
4	short	5	80	2	0.08	0	6
4	long	10	40	1	1	49	40
4	myopic	10	40	1	0.08	0.25	6
4	short	10	40	1	0.08	0	6

Table 2.1: State of the network depending on the capitalization ($e = 2.5, \beta = 0.98$)

their investments to $85 < A_{\max} = 100$ investments, to reach the optimal payoff. The equilibrium values of the model are $G^* = 2.1$ and $V^* = 6.3$. This value also minimizes the number of contaminated defaulting and strategically defaulting agents. For any value of $A > 85$ there is a net positive probability of systemic risk, as explained in the previous section. For this reason, the regulator may decide to reduce the maximal number of investments of the agents. By decreasing the maximal number of investments: from 100 when $\phi = 5\%$ to 50 when $\phi = 10\%$, the regulator also decreases the expectations of the maximal gains of agents: from

$G_{\max} = 2.5$ to $G_{\max} = 1.25$. This also decreases the maximal value of V^*: $V^* \leq 3.75$. This situation creates more strategically defaulting agents. To avoid the risk of contagion, agents decide to make less investments: $A^* = 9$ than the maximal number allowed: $A_{\max} = 50$. This situation illustrates that, when the regulator strenghten the prudential ratio to avoid the systemic risk, and agents are myopic, liquidity may dry up. Mechanically, if the capitalization of agents is decreasing, due to an external shock, this could produce the same effects; by reducing the anticipations of the expected payoffs of the agents.

- When agents optimize over their infinite lifetime, even with low capitalized agents, the use of a prudential ratio to avoid major defaults is necessary and successful.
- When agents optimize over the current period only, the use of a prudential ratio is useless, it cannot reduce the number of strategically defaulting agents. Agents make less investments than they could. Defaults do not propagate.
- When agents are myopic[a], the use of a prudential ratio depends on the capitalization of agents in the network:
  - necessary and successful for highly capitalized agents, prevents systemic risk,
  - useless and harmless for weakly capitalized agents, this is the equivalent of one-period optimization,
  - necessary but counterproductive for intermediately capitalized agents: prevents systemic risk, but generates liquidity problems.

[a]For risk-averse agents, depending on the concavity of the utility function, V^* would be reduced, and this would accentuate the myopic behavior of agents

2.5.4 Conclusion

The model of network developed in this piece research has several common implications with the recent literature: the risk of default is less important for very low and very high connected networks. Nevertheless, we must handle with care definitions of low and high connectivity, and incomplete, sparse or dense networks, because these notions vary a lot depending on the network context: a complete network in percolation models (Anand et al., 2012) would be an incomplete network in contagion theory (Allen and Gale, 2000), Figure 1.3. Our range of connectivity starts with the lowest percolation threshold (1 investment starting from each agent, corresponding to the lowest bound of Watt's global cascade window Figure 1.6) and reaches high connected networks ($\frac{C}{\phi} >> 1$ investments by agent), which remain still less connected than the complete network (Figure 1.2) of contagion theory... Apart from that, our threshold of systemic risk (a large number of investments by agent) does not correspond to the usual percolation threshold (a few investments by agents). Indeed, the threshold of systemic risk mainly depends on the capitalization and the expectations of agents.

Systemic risk exists when agents are myopic, and is increasing with respect to the connectivity. The model has new features, such the endogenous shocks, represented by the strategically defaulting agents. The model describes in a simple way the effect of agents' choices on the network and the effects of the network structure on the financial health of agents. The model also takes into account the horizon perspective to determine the optimal behavior of agents. The model derives policy implications about the use of a capitalization ratio. To conclude, capitalization ratios must be wisely employed, especially when confidence on financial markets is threatened, and when agents tend to adopt myopic behaviors, because prudential ratio, in these situations, may barely trade liquidity crisis for a systemic risk.

Appendix B

About the computing

To obtain the simulations of Chapter 2, we have been programing on Matlab. After choosing an arbitrary number of agents n, we had too fill the M matrix corresponding to the financial network, such that, for any number of asset by agent, conditions (2.4) and condition (2.5) are satisfied. For a number A of investment by agent, we construct recursively the matrix M. We add one investment for each agent, then 2 investments for each agent... until A investments by agent. But the investments are not purely random to satisfy (2.5). In the algorythm, when adding a new investment for all the agents, this requires to create a list of the empty elements of the M matrix, to look for the agents which need an investment to satistfy (2.5). This step was not obvious because it also needs to look for the agents which can make an investment, and which have not already invest towards this agent. To verify that the matrix is properly filled, we sum the lines and the columns of the positive/negative part of the matrix.

Once the matrix M is filled, we can find the agents which initially strategically default, those such that their net number of investments (received minus issued) exceed the discounted sum of maximal payoffs. Then we calculate using the matrix the effect of these initially strategically defaulting agents, this gives some contaminated defaulting agents. With these two values of strategically defaulting and contaminated defaulting agents, we calculate the new discounted sum of maximal payoffs, and the new number of strategically defaulting agents, and so on. This is possible using a large number of loops.

There is another problem, it is possible to calculate the individual payoffs of all the agents of the period, but difficult to estimate correctly the value V^* of the expected discounted sum of incoming payoffs. This required to try (guess and verify method) a large number of values of V^* and select the one which corresponds to the average payoff for a large number of networks. The simulation of the one period network is a lot easier, because $V^* = 0$, as explained in section 2.4. A simulation of one network of 400 agents with the capitalization $C = 10$ and $\phi = 10\%$ takes about 10 minutes on a modern 2 cores 4 threads processor in parallel computing. To get smooth curves, we must simulate at least 2000 networks, which lasts for 340 hours, about 14 days nonstop. In addition, we had to test a lot of different parameters' values.

Chapter 3

Bubbles in asset prices

Contents

Inspired by a recent article, Miao and Wang (2011), we would like to exhibit the possiblity of bubbles on asset prices in a production economy. When bubbles exist, which conditions may help them to grow or burst? In that direction, do interest rates impact these "bubbles", and to what extent? To answer these questions, we start from the first modern conception of bubbles, (Blanchard and Watson, 1982) to reach recent evolutions of this problem. Preceeding crises, one often explains that there exists a bubble in the economy. As mentioned in the introduction, it seems that the subprime crisis was carried by a U.S. real estate bubble. We want to understand how research has been able to generate bubbles.

This chapter presents the context of the following research through a selection of major research articles. Of course, the literature on bubbles is quite developed and therefore it would be unrealistic to try to be exhaustive. The selection of the following articles is designed to present the main evolutions of the analysis of bubbles, especially related to the following work[1].

3.1 How to introduce bubbles in prices?

Historically, bubbles have occured frenquently enough and deserve to be explored. There exists an abundant descriptive literature on bubbles. For example Garber (1990) is interested in the Dutch tulipmania (1634), the Mississipi (or Compagnie des Indes) (1719) and South Sea (1720) bubbles. There are explanations about the market fundamentals and the expectations of participants: the scarcity of "sick" bulbs for the tulipmania, the hope of profitable quick growth for the Mississipi Company, and information asymmetry about national debt consolidation and hope of profitable trade in the South Sea Company. All these situations happened because buyers were expecting to make (large) profits, and the increase of the prices was therefore reflecting anticipations of higher yields, and higher prices. We do not focus on the descriptive articles on bubbles. In addition, there is a large number of models accounting for large variations in asset returns, e.g., Sornette

[1] To improve the readability of the chapter, we adopte the same notations to design the same objects, even if they do not correspond to authors' original printings.

(2009), which provides an in-depth survey of recent models derived from complex system theory. Sornette notably shows that classical models (e.g., Garch, Gaussian walks, linear models) fail to predict crises correctly, while other models (rationality with imitations behaviors, non-linear correlations, power laws...) are a lot more reliable. We start with the theoretical literature which model the idea that current prices reflect anticipations of incoming profits.

3.1.1 First characterization of bubbles

As explained by Blanchard and Watson (1982), the price of an asset should reflect market fundamentals under rational behaviors and rational expectations of agents. Precisely, the price of an asset should depend exclusively on the information available about its current and future returns. If the price differs from its fundamental value, it means that there exists a rational deviation that we can name "bubble". Blanchard and Watson (1982) restrict their attention to the existence of rational bubbles because irrational ones might be more difficult to conceptualize; even if a large number of studies, especially in the experimental literature, tend to prove the irrationality of agents, (Lei, Noussair, and Plott, 2001).

The yield R_t of an asset A whose price at time t is p_t and dividend of time t is d_t verifies:

$$p_t R_t = (p_{t+1} - p_t) + d_t. \tag{3.1}$$

With market clearing condition, someone holds the asset. With the no-arbitrage condition and rational expectation, the expected yield of this asset must correspond to the interest rate r of the riskless bond, otherwise no one would hold the asset. Given the information available at time t, designed by Ω_t, the yield R_t of the asset satisfies the following relation[2]:

$$\mathbb{E}[R_t|\Omega_t] = r. \tag{3.2}$$

The two previous equations give a dynamic system. Blanchard (1979a) shows that, using the property of induction on expectations, because $\Omega_t \subseteq \Omega_{t+1}$, the

[2]This relation can also be interpretated as the existence of at least one risk-neutral probability.

system can be solved recursively forward. One of the solutions to this system at time t is:

$$\boxed{p_t^* = \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^{i+1} \mathbb{E}[d_{t+i}|\Omega_t]. \qquad (3.3)}$$

This particular solution is called the fundamental price, it corresponds to the present value of the expected dividends. Let us consider any sequence (b_t) that satisfies:

$$\mathbb{E}[b_{t+1}|\Omega_t] = (1+r)b_t. \qquad (3.4)$$

Then the price $p_t = p_t^* + b_t$ is also a solution to the same problem. The component b_t does not need to be related to the dividends. In this sense, b_t is called a bubble. Since $(1+r) > 1$, the bubble component is growing over time. If $b_t > 0$, the price of the asset $p_t^* + b_t$ diverges in the long run. This seems to make deterministic bubbles irrational. As we will see, this is a particular case of the transversality condition, which indeed rules out any bubble satisfying equation (3.4).

Blanchard and Watson (1982) introduce more realistic bubbles by making them stochastic. Let us consider a particular bubble (b'_t) which can either exist or collapse by introducing a probability θ:

- $b'_t = \frac{(1+r)}{\theta} b'_{t-1} + \mu_t$ with a probability θ,
- $b'_t = \mu_t$ with a probability $(1-\theta)$, (3.5)

where μ is a random variable such that $\mathbb{E}[\mu_t|\Omega_{t-1}] = 0$. This bubble b'_t keeps growing with a probability θ or crashes with a probability $1-\theta$. It still satisfies $\mathbb{E}[b'_{t+1}|\Omega_t] = (1+r)b'_t$ and therefore the sequence $(p_t + b'_t)$ is also solution to the problem. We remark in this example that the growth rate of the bubble – when it keeps growing – is $\frac{1+r}{\theta} > 1+r$, this compensates the risk of crashing. As the authors explain, a lot of different bubbles are possible, making varying θ over time, and even over the time duration of the bubble for example, also detailed in Blanchard (1979b). This is also possible to link the bubble to the fundamental value, for example by correlating the dates of crashes of the fundamental price and the bubble. In addition, since the bubble adds risk in the price, if agents are risk averse, the growth rate of the bubble has to be even higher to compensate the possibility of crash: the probability of crash represents the risk associated to the bubble, and therefore, when the bubble exists, it has to yield more than the risk free asset.

Blanchard and Watson (1982) raise the question of the possible influence of bubbles on fundamental prices. They provide examples on housing prices or even on firms' equities: the existence of a bubble increase the overall price and might reduce the dividends in the following, through decreasing productivity or increasing stocks. As a consequence, the fundamental price of the asset keeps decreasing while the bubble keeps growing, until it perhaps bursts. The remaining part of the article is an empirical approach of the American Stock market using the variance and conditionnal variance of prices to check wether bubbles may not exist. Their analysis prooves that bubbles on asset prices are likely to exist.

Sornette and Malevergne (2001) generalize this model by looking at the tail size distribution of such stochastic bubbles, on equation (3.5). Using the no-arbitrage condition, they show that the tail of the return distribution of the asset is hyperbolic with an exponent less than 1. When looking at multidimensional process, they find that tails of the return distribution of the assets follow power laws, with the same asymptotic exponent. Unfortunately, they show that this theoretical exponent disagrees with the empirical estimates.

From there we adopt a quite standard definition of bubble: when the price of an asset exceeds the theoretical sum of the stream of dividends also called fundamental price, the remaining part is called a bubble.

3.1.2 Linearity of bubbles?

Using the formulation of Blanchard and Watson (1982), Gilles and LeRoy (1992) went into the mathematical characterization of bubbles in depth. Let us consider an asset A which delivers the stream of dividends $(d_i)_{i>0}$ discounted at the interest rate r. Assume that the price of the asset A in period t can be written the following way:

$$p_t(A) = \sum_{i=t+1}^{\infty} (1+r)^{-(i-t)} d_i + b(1+r)^t. \tag{3.6}$$

For any $b > 0$, there is a bubble in the price of the asset A, in addition to the fundamental price. Let A_t be the same asset as A but starting only at time t. Before time t the dividends of A_t are 0. We bring back the valuation to the time 0 for the asset A_t:

$$p_0(A_t) = \sum_{i=1}^{\infty} (1+r)^{-(i+t)} d_{i+t} + b. \tag{3.7}$$

Using the property of linearity of the valuation, Gilles and LeRoy (1992) deduce that the bubble b is also linear with respect to the asset: for any constant k, $b(kA_t) = kb(A_t)$ and $b(A1_t + A2_t) = b(A1_t) + b(A2_t)$. For example, a bubble can be a function of the stream of dividends of the asset expressed at time t as follows:

$$b_t(A_t) = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} d_{t+i}}{n}. \tag{3.8}$$

Instead of considering the asset A, let us consider the sequence of assets A_i such that A_i delivers the same dividend as A only in period i, and 0 otherwise. Using equation (3.8) the bubble component $b_t(A_i) = 0$ for all A_i.

The property of countably additivity, which says that the value of the sum is the sum of the values of the parts remains true only for the fundamental price. For the bubble on the opposite, the value of the infinite sum must exceed the values of the parts. Precisely, considering the space of bounded sequences, a bubble corresponds to a mathematical "pure charge", which is not countably additive, whereas the fundamental value is, also called a "measure". Gilles and LeRoy (1992) consider economic problems where bubbles may occur. Though they exhibit special cases of utility functions and endowments where bubbles exist, they also show that when the utility function discounts the future, no bubble is possible on bounded cash flows. As a conclusion, bubbles are rare when usual hypotheses are employed.

Many authors have been looking for bubbles in usual economic and financial models: either models with infinitely-lived agents (traders), or models with overlapping generations.

3.1.3 Myopic behaviors allow for bubbles

Tirole (1982) studied the case of purely speculative markets. This is indeed a particular case of Blanchard and Watson (1982). In a purely speculative market, traders exchange claims for each asset at an equilibrium price before knowing the realization of the price of the asset, which is indeed a random variable. The gain of trader i depends on the net quantity x_i he trades, the equilibrium price p and the ex-post realized price $\tilde{p}$. In a "static speculative market", the total gain is 0 and there is no insurance. Traders maximize their utility with respect to their information sets, given market prices. Traders are also supposed to have the same probabilities (priors) on the set of states of nature and act under a rational expectations equilibrium. In this case, concave utility functions, from risk neutral to risk averse, rule out any possibility of speculation. Risk averse traders do not trade because the expectation of the gain is 0, only risk neutral traders can trade. Speculation and bubbles may exist if there is at least one irrational trader (wrong anticipations, systematic biais or anticipations depending on irrational beliefs) or a risk loving trader. Tirole extended the case to a dynamic sequential stock market with assets delivering a sequence of dividends. Each trader at each period has a signal about the stochastic process followed by the dividend. In the case of homogenous information between traders, they are used to compare the current price and the expected future price given by the probability distribution of it. Given this reasoning they choose the action, buy, sell, or do nothing. In this context, Tirole first deals with "myopic rational expectations equilibriums". In this equilibrium, information is heterogenous between traders. An equilibrium is the set of traders' price forecast functions, a set of asset holdings x_t^i, market clearing ($\sum x_t^i = 1$), and short-run (refering to myopic) maximization behaviors of traders as explained by the following equations. Indeed, the ability to sell the asset at the following period may influence the price, because the current price depends on the next period price or the whole sequence of dividends.

- If short sales are allowed: $p_t = \beta \mathbb{E}_t[d_{t+1} + p_{t+1} | \Omega_t, \omega_t^i]$, where Ω_t is the common information, ω_t^i the signal of trader i and β the discount parameter.

- When short sales are impossible, then the holdings depend on the forecast prices:
  - $p_t = \beta\mathbb{E}[d_{t+1} + p_{t+1}|\Omega_t, \omega_t^i]$ then $x_t^i \in [0, 1]$.
  - $p_t > \beta\mathbb{E}[d_{t+1} + p_{t+1}|\Omega_t, \omega_t^i]$ implies $x_t^i = 0$.
  - $p_t < \beta\mathbb{E}[d_{t+1} + p_{t+1}|\Omega_t, \omega_t^i]$ implies $x_t^i = 1$.

Tirole showed that the formulation $p_t = \beta\mathbb{E}_t[d_{t+1} + p_{t+1}|\Omega_t, \omega_t^i]$ remains exact even if short sales are forbidden. Then Tirole also proved by induction that for active traders in a finite horizon problem T the price only depend on the dividend forecast sequence[3], bubbles do not exist:

$$p_t = \mathbb{E}\left[\sum_{k>1}^{T-t} \beta^k d_{t+k}|\Omega_t, \omega_t^i\right]. \tag{3.9}$$

On the opposite, when the time horizon is infinite, prices may include bubbles components satisfying a martingale property, wether short sales are allowed or not: $\forall l, i, t$

$$B_t(\omega_t^i, p_t) = \beta^l\mathbb{E}[B(\omega_{t+l}^i, p_{t+l})|\Omega_t, \omega_t^i]. \tag{3.10}$$

When bubbles arise, they have the same values for all market traders if the information is homogeneous: $\forall i,\, \omega_t^i \subseteq \Omega_t$. When the time horizon is infinite and the information is homogeneous, a bubble is no longer possible on a finite set of traders because it would require that each trader realizes his gains in a finite time and therefore sells the asset at some time, which violates the market clearing condition: at each period every asset has to be held by at least one trader.

The last step of Tirole's reasoning was to study a fully dynamic rational expectation equilibrium. The difference between this model and the myopic equilibrium

[3] also refered as market fundamentals, in opposition to the expected short resale price, which might be "disconnected" from fundamentals.

is the maximization process of each trader: traders maximize their expected present discounted gains from the current period to infinity using the common prior Ω_t and their personal informations ω_t^i. This new maximization process restricts the possible sets of prices and rules out bubbles when the number of traders is finite. Tirole proved that bubbles are impossible with fully dynamic rational expectations. Rational expectations models are widely used in finance and economics because of their high plausibility and technical handiness, on the opposite, irrationality of traders, unusual structures of information, asymmetry, sunspots, are difficult to model. Adaptative – multiple – anticipations produce irregular equilibrium prices, as shown by Brock and Hommes (1997). Also, behavioral finance is dealing with those problems. For an example about heterogenous information, and different behaviors of traders, see Frinoa, Johnstone, and Zhenga (2004). When agents adjust their over-optimistic expectations on prices, this generates super-exponential bubbles (Hüesler, Sornette, and Hommes, 2013). We remain in the context of rationality to deal with the existence of bubbles.

Going back to myopic behaviors, bubbles are only possible when the number of traders is infinite, the time is infinite, and when traders get their gains on bubbles in the short term. Overlapping generation models satisfy these conditions. Tirole (1985) clarified the role of money in those economies starting from the framework of Diamond (1965). The fundamental market value of the money is zero, because it does not deliver any dividend. As soon as money has a non-zero price, it is a bubble. By arbitrage, money must have the same expected yield as capital, so the price of money in perfect foresight (also called rational expectations) must verify:

$$p_{t+1} = \mathbb{E}[\frac{1 + r_{t+1}}{1 + n} p_t | \Omega_t]; \tag{3.11}$$

where r_{t+1} is the yield of capital from time t to time $t + 1$, n is the growth rate of the number of persons in the economy and p_t the price of money at time t. The existence of a bubble requires an equilibrium in which the sequences of interest rates, wages, savings among which bubbles (money) are such that the market clearing conditions are satisfied and a non-zero sequence of bubbles exists, especially in the long term. The results of Tirole's analysis can be understood by looking at the equation (3.11). In the Diamond's model, an equilibrium interest rate r^* is deduced from the wage and the market clearing conditions. If this equilibrium interest rate exceeds the growth of the population, there is a unique equilibrium without bubble. The interest rate r_t converges to the equilibrium one r^*. Indeed the bubble would grow faster than the resources of the economy, as showed by the equation. On the opposite, for the other cases, bubbly equilibria exist and the value of the bubble mainly depends on the value of the equilibrium interest rate r^*. If $r^* > 0$, depending on the initial values of the model, either the economy reaches a bubbleless equilibrium in the long run, or the bubble converges to a non-zero value and the interest rate r_t converges to the growth rate of the population n. In the equation (3.11), it means that the price of money is constant. When the equilibrium interest rate is negative, there only exists one equilibrium including a persistent bubble and the interest rate also converges to n. In the following, Tirole also proved that bubbles can still exist if there are also rents in the economy[4]. The existence of bubbles depends on the equilibrium interest rate of the economy, the initial conditions, and the parameters values of the model, the form of utility functions. Tirole concluded that the existence of a bubble can be an efficient equilibrium in the intermediary case. Indeed, raising the question of the efficiency of bubbles is relevant, especially when two or more equilibria coexist. Recently, Farhi and Tirole (2013) developped an Overlapping Generation Model with firms where liquidity scarcity may generate bubbles that might be dynamically efficient: the level of capital is larger than the level provided by the golden rule. Martin and Ventura (2012) introduced an "investor sentiment" in this overlapping generation model (Tirole, 1985) which generates periods of bubbles, overall growth and consumption while a change in this sentiment produces a collapse of these bubbles.

Again, many studies which reveal that, under weaker assumptions on the agents, there might exist a large number of equilibria, among which inefficient ones. For example, Yukalov, Sornette, and Yukalova (2009) analyze the price of an asset traded by a population of heterogenous agents with uncertainty: there exists a large

[4]The existence of bubbles in overlapping generation models when borrowing is allowed was recently undertaken by Hillebrand (2012).

number of equilibria and the dynamic of the price of the assets switches between different regimes.

Abreu and Brunnermeier (2003) deal with rational traders (arbitrageurs), but point out coordination and information problems. They shows that a bubble may persist, because traders do not coordinate. Traders know that there is a bubble in the price because they receive information sequentially: this means that traders do not receive informations on the fundamentals at the same time. They want to benefit from the bubble before it bursts, therefore they keep buying the overpriced asset. When the dispersion of opinion is sufficently large, the bubble never bursts. When the dispersion of opinion is smaller, the bubble bursts before reaching its maximal value, because traders coordinate. Public information also helps them to coordinate and therefore provokes crashes. This model shows that when information is not common, bubbles at equilibrium are likely to last.

Santos and Woodford (1997) formalize the general case therefore including the two previous papers of Tirole. They point out the difficulty of ruling out bubbles when the time is infinite. Their framework mixes spot markets for goods and securities for a infinite sequence of dates in a intertemporal general equilibrium model. We do not detail their formalization because of the complexity of this article. They establish a strong theorem which states the existence of a state price process that reaches the market prices, even when the market is incomplete, for any security either of finite maturity or in positive net supply. They require that consumer's preferences are increasing for all goods and stricly for at least one, and very tight borrowing limits: at least one consumer has to be able to repay another one's future endowment for some state price process. When they tighten the preferences of the consumers by introducing discounting or impatience in the consumer preferences, they get a stronger version of the prices in terms of one state price instead of any possible state prices. In this situation, no bubble is possible anymore with respect to the chosen state prices. When there exists a portfolio such that, at any point in time it yields more than the aggregate endowment of that state, then there is no need for borrowing constraint for the conclusions to hold.

As the authors explain, the usual well-known models that allow for bubbles violate at least one of the hypotheses.

3.1.4 Infinite horizon: the transversality condition against bubbles

The well-known transversality condition is widely employed to solve infinite horizon models, for example to solve Bellman equations. But this is also a formidable protection against bubbles, detailed for example in Obstfeld and Rogoff (1983). Note that the transversality condition requires the utility function to be differentiable. To show how it works, let us consider a simple example. Suppose we want to solve a standard maximization problem of a househould whose utility function is u, stream of consumption $(c_t)_{t\geq 0}$ and discount rate β. The household owns a share A_t of a firm, which price is p_t and which pays y_t at each period. Given an initial wealth $A_0 = 1$ and a resource constraint linking the share A_t, the price of the firm p_t and the consumption c_t:

$$p_t(A_{t+1} - A_t) + c_t = A_t y_t,$$

we solve:

$$\max_{c_t, t\geq 0} \sum_{t\geq 0} \mathbb{E}_t[\beta^t u(c_t)], \tag{3.12}$$

where $\mathbb{E}_t$ denotes the expectation conditional on the information available at t. We impose the usual Inada conditions to ensure the existence of a solution: $u(0) = 0$, $u' > 0$, $u'' < 0$, $\lim_0 u' = \infty$ and $\lim_\infty u' = 0$. An equilibrium price sequence is such that the market clearing conditions are satisfied: $\forall t$, $A_t = 1$ and then $c_t = y_t$. From this, we deduce the first order Euler equation:

$$u'(y_t)p_t = \beta u'(y_{t+1})(p_{t+1} + y_{t+1}). \tag{3.13}$$

From this recursive equation on prices, and given the sequence of incomes the firm delivers, we can deduce the general form of the equilibrium price:

$$p_t = \mathbb{E}_t \left[\sum_{k \geq 1} \beta^k \frac{y_{t+k} u'(y_{t+k})}{u'(y_t)} + \lim_{k \to \infty} \beta^k \frac{p_{t+k} u'(y_{t+k})}{u'(y_t)} \right]. \tag{3.14}$$

The term $P_t = \mathbb{E}_t \left[\sum_{k \geq 1} \beta^k y_{t+k} \frac{u'(y_{t+k})}{u'(y_t)} \right]$ corresponds to the fundamental value of the asset. If we consider a sequence q_t such that $u'(y_t) q_t = \beta \mathbb{E}_t[u'(y_{t+1}) q_{t+1}]$, we see that $P_t + q_t$ satisfies the Euler equation (3.13) as well as the general form of the equilibrium price (3.14). The transversality condition of this problem can be written as:

$$\lim_{k \to \infty} \beta^k u'(y_k) p_k = 0. \tag{3.15}$$

This transversality condition imposes $q_t = 0$ and the general form of the equilibrium price is restricted to its fundamental value. Imposing the transversality condition rules out bubbles in most general equilibrium models, and together with the Euler equation it is a sufficient condition for the optimality of the solution, as explained in Stockey, Lucas, and Prescott (1989). The transversality condition can be understood as a first-order condition written “at the end of time” of the problem. It is sufficient but also necessary in a finite time problem, because it is the first-order condition for the last term. The transversality condition in the inifite time problem is the limit for $t \to \infty$ of the transversality condition of the finite time problem. It means that the value of the wealth – in this problem p_t – must not widely exceed its discounted utility. It prevents the accumulation of wealth over and over. We can conclude that there will not exist any sequential dynamic infinitely lived agent discounting optimization model in which bubbles could emerge in the presence of the transversality condition.

3.2 Collateral constraints responsible for bubbles

In the following of this review, some models do not deal with financial assets and their dividends. For example, they may consider the land, its price, and its production. In accordance with the authors and the topics, we keep the original formulations of the problems. Actually, any production factor, (land or firm) can be interpretated as an asset, and the production per period (output) can be interpretated as a dividend of the asset. As a consequence, the concept of bubble can be transfered to non-financial subjects. There is a limit to this analogy. For example, Bouaskera and Prigent (2008) show that firms, and more broadly goods and services, cannot be valued exactly as financial assets. Because the investment decision is irreversible, the price of a firm is also depending on market's features, such as the global demand. In the sequel of the dissertation, restricting assumptions on the market demand still allow to hold the analogy in the value of firms and assets to be true.

3.2.1 Prices of land

The framework we use in our research was initially explicitely introduced by Kiyotaki and Moore (1997). Indeed, their paper does not deal with bubbles, but more with credit, borrowing limits, and their effects on the economy, especially on prices. It remains useful to include it in this presentation because it explains the genesis of some of our assumptions, even though all of them are not used for the same purposes. The authors show how credit may affect the economy through a model of lenders-borrowers in a production economy with perfect foresight. They start from a fixed supply of land K which is the unique production factor. This "funny" context of land, farmers and vegetables can be interpretated as well in a context of capital, firms and goods. On this land there are two types of producers that do not have the same production function. They produce a unique tradable good that they also consume. Among them, farmers have a linear production function $K \to aK$, while gatherers have a marginally decreasing production function $G(K)$ with $G' > 0, G'' < 0$. In addition $a > G'(0)$: as we understand, farmers have a better production function, (this assumption is justified by their specific skills). Both farmers and gatherers borrow to buy the quantity of land they use at the market price p_t at each period. They take one-period loans b_t and repay them with the interests on the loan at the end of the period $R_t b_t$. The interest rate is supposed to be constant to R. They face at each period a budget constraint which takes into account their production, their consumption c_t, their level of debt b_t, and the land they use k_t at the price p_t. For example, a farmer faces the following constraint:

$$p_t(k_t - k_{t-1}) + Rb_{t-1} + c_t = ak_{t-1} + b_t. \tag{3.16}$$

The two types of producers are risk neutral and maximize their discounted consumptions over their infinite lifetime:

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty} \beta^s c_{t+s}\right]. \tag{3.17}$$

Because they own specific skills that give them a better production, the farmers are subject to a borrowing limit: they have to be able to repay the debt with the only asset they have, the land: $Rb_t \leq p_{t+1}k_t$. This constraint is designed to capture the fact that, if a farmer stops working, nobody will be able to produce any good on that land, because of the specific skills he owns. As a consequence the lenders collateralize the debt on farmers with the value of the land. All these assumptions are the keys of the model, though the authors add some others to ensure the existence of a competitive equilibrium. This competitive equilibrium is a sequence of quantities of land, consumptions, debts, and land prices such that the producers and gatherers maximize their utilities, markets clear on land and consumption, and debts clear. At the equilibrium, the authors prove that farmers are net borrowers and gatherers are lenders. To kill bubbles on land prices, they impose the transversality condition.

The first results are not intuitive: when the prices of land climb, the global demand of farmers increases because they can borrow more given the collateral constraint. Even if they have larger repayment due to the increase of the debt, the net worth of the farmers increases more than the debt because of the leverage effect. The debt constraint reduces the quantity of land that farmers would get in an unconstrained situation, because it would equalize marginal productivity of farmers and gatherers. Particularly, the effect of an unexpected productivity shock is highly amplified by the debt constraint. There is a intratemporal static effect increasing the prices of land because the demand of farmers increases and there is overall a persistent effect due to the changes in prices of the land. That would have not happened if there was no debt constraint. The respective multipliers' sizes highly depend on the interest rate, which is reliable to the discount rate of the gatherers. However, the global response of the economy to such a shock is really large due to the ratio debt-asset of the farmers, which is quite large. To improve the efficiency of the model, the authors introduce the technique of the farmers in the model, such that farmers do not invest only in land as before, but also invest in technique at each period, because it depreciates. The technique itself has no liquidation value. Again they add a couple of parameters' hypotheses to ensure the existence of an interesting competitive equilibrium. When they study the effects of an unexpected productivity shock, the economy now creates cycles. When the demand of farmers in land increases, the level of debt also increases, but the increase of the debt limits the farmers' ability to buy land. This creates large and persistent oscillations, also called cycles. To get even better results, the authors add a probability of investment: there exists a probability such that a farmer will not be able to invest at each period, especially in technique. This is supposed to represent difficulties or the fixed cost that usually restrict investment possiblities of firms. When a farmer is unable to invest, his technique depreciates, and so the farmer can only use a smaller part of the land he owns, and therefore he sells the remaining part. From the linearization around the steady state, adding technique in the economy influences the prices more than the quantities, and the probability of investment makes the cycles last over more periods. From these conclusions, we may remember that credit constraints on a production economy make it more responsive to production shocks and that restrictions on investment make these effects last more than usual.

Bernanke and Gertler (1989) also analyzed how collateral prices influence the real economy, amplifying shocks. Their model is based on a neo-classical approach, in which entrepreneurs are also borrowers. Their limits of borrowing depend on their balance sheets, and the agency costs are also inversely related to balance sheets of the borrowers. The result is the amplification of productivity shocks in both directions, either the economy is accelerated when the balance sheets are good, or the credit is limited and the agency costs make the investment more difficult.

This idea of considering asset prices not only as values but also as credit collateral expanded quickly. The credit helps the growth of the asset that is also collateral, this self-acting phenomenon may help emerge distortions in prices among which bubbles. For example, subprime loans were mortgage loans, that does not help stabilize prices when demand is crashing. In a recent paper, Kunieda (2008) extended Tirole (1985) by adding borrowing and analyzed how bubbles on money can be inefficient, and what government may do to correct the inefficiency.

3.2.2 Bubbles in firms' prices

Kocherlakota (2009) developed a model where limiting the debt to the value of land generates a bubble on the prices. We will detail this model, because it uses the framework of Kiyotaki and Moore (1997) and deals with bubbles in this context. The economy is composed of infinitely lived entrepreneurs who use land to produce a common good. At each period, only a fraction of them face investment opportunities and the others do not. Heterogeneity among entrepreneurs is created with the production function: $y_t = A_t k_t^{\alpha} n_t^{1-\alpha}$, where y is the prodution, k the capital, n the labor and A the technological factor which takes values either 0 with a probability $1 - \pi$ or 1 with a probability π: A_t is a random value i.i.d over entrepreneurs and time. The value of A_t is known at the previous period $(t-1)$. When it takes 0 value, the entrepreneur does not want to produce anything, and on the contrary when it takes value 1 the entrepreneur is willing to invest to produce. To achieve the investment opportunities, money needs to be reallocated, and this is done through loans. A new feature of this model is to endow each entrepreneur with one unit of land at the beginning. When they borrow, entrepreneur's loans are secured by the value of the land they own. Since land does not deliver any financial stream, any positive price on the land can be interpreted as a bubble. Two equilibria exist in this model: either there is no borrowing and the price of land is zero, or money is actually reallocated through loans, and the land has a positive price. As in the previous models, an equilibrium is a sequence of prices of land, wages, interest rates of loans, and a sequence of consumptions, units of land, loans, capitals, units of labor such that, given the prices and the budget constraints, the entrepreneurs maximize their utilities and markets clear. Since there is no constraint on the amount of work; entrepreneurs that produce choose the amount of work they need to maximize their profits, and therefore they all have the same capital labor ratio, an hypothesis that we will also use in our models. This delivers the equilibrium wage. Given the two equilibria, the author compares the levels of wealth, and proves that there is more wealth with bubbles, the consumption, output, and wages are higher in the bubbly steady state, i.e. when land has a positive price. After describing these equilibria, Kocherlakota generates a bursting bubble and its effects on the economy. This is designed to capture housing prices in the United States, and their links with loans. Starting from a bubbly equilibrium, in which land has a non zero price, and where money is reallocated through loans, the author introduces a probability of switching from this bubbly state to the non bubbly one, less economically efficient. When the economy reaches the no bubble equilibrium, it cannot go back to the other state. In the initial state, the economy is bubbly and the equilibrium of this stochastic model approaches the bubbly one. When the bubble bursts, the net borrowers are not affected, but the net lenders are greatly affected, the distribution of wealth changes. The average level of capital and wages decrease, as well as consumptions, because the land value is destroyed. When the bubble collapses, the lenders always lose money, because their loans are not refunded, while borrowers may benefit from the bubble's collapse because they do not have to refund their loans, unless the depreciation rate of capital is too high and they also lose. The analysis reveals that even a stochastic bubble increases the amount of capital and the wages before the bubble bursts, and therefore a stochastic bubble has a net positive impact on the economy. Kocherlakota raises the question of the role of the government, especially his ability to secure loans because they are risky in the presence of stochastic bubbles. If the government secures the loans by providing bonds, optimal response of the agents would be to hold as much as they can of these bonds, and there is no easy way to see how the government could inject such an amount of cash when the bubble bursts. The only possibility for the government would be to secure its own debt by taxation and therefore substitute for private sector bubbles. In this model, the need of reallocation of capital and borrowing constraints have been able to create bubbles on the price of land, which generates more output, consumption and wealth.

Miao and Wang (2011) consider a production economy where the values of firms themselves are collateral for borrowing. They show that investment restrictions associated to investment difficulties generate bubbles. We choose to adopt this approach mixing a production economy, possible restrictions in the investment and credit constraints. We do not detail the model in this section because it will be fully analyzed in next chapter.

Kunieda and Shibata (2012a) propose the same kind of framework to create a bubble. Agents have the same production function, but there exist idiosyncratic shocks on the production function: $y_t = A\Phi_{t-1}k_{t-1}$, where y_t represents the production, k_{t-1} is the capital of the previous period and Φ is a productivity function and random variable i.i.d over agents and time. Agents consume at each period and have also access to an asset b_t that they can use either as deposit, or as debt, subject to the interest rate. The agents maximize the expected utilities of consumption over their lifetime, with respect to a usual budget constraint. As in other models, borrowing is limited to a fraction of the net worth of each agent. At each period, an agent is willing to produce if his production function brings more than the rent, which delivers a condition on his value of Φ. Besides this production sector, there exist financial intermediaries that lend to agents who need it and also collect the deposits from the others. These financial intermediaries have also access to an "intrinsically useless asset" because the deposits exceed the debts given the borrowing constraint. The supply of this new asset is fixed and the asset is priced at p_t. As soon as this asset is priced, it is a bubble, by definition. By arbitrage, the price of the asset must verify $\mathbb{E}_t[p_{t+1}] = \mathbb{E}_t[r_{t+1}]p_t$, otherwise no financial intermediary would buy it. The authors aggregate all the variables depending on the investing agents and the non-investing ones to get the dynamics of the intrinsically useless asset, especially as a function of the distribution of Φ. They get the global equilibrium and the existence of the useless asset depends on the distribution of Φ: either the debts correspond exactly to the deposits, and the asset does not exist, or the asset covers the difference and has a positive price. It might happen that two equilibria coexist, one without bubble and one with the bubble. The authors prove that the growth rate of the economy is larger in the bubbly one. Precisely, the bubble increases the interest rate, and this reduces the number of investors, among them the less productive ones. This is what they call the "crowd-out" effect of the bubble. On the opposite the net lenders get a higher yield on their deposits, and this gives them more liquidity if they become investors in the following periods. This is the "crowd-in" effect of the bubble, which dominates the other. When there is no bubble, the less productive agents – having a low Φ – will get low production and this limits their ability to invest, even if they become high productive agents in the following. The authors prove that when both equilibria exist, the bubbleless one is dynamically inefficient whereas the bubble can correct that. From these results they develop then a Markov process acting on the variable Φ to model a sunspot equilibrium. In this situation there are two interest rates depending on the state in which the economy is located. One of the states is bubbleless with a low growth rate, and the other is bubbly. When the economy switches from the high growth rate state to the low one, this is irreversible, it can never reach again the high growth rate state. This switch can be interpreted as a financial crisis and the intrinsically useless asset does not exist anymore. The authors suggest in the following that the governement could back this asset to keep the economy in the efficient bubbly equilibrium by taxing agents' net incomes. In this case, the bubbly steady state becomes the unique perfect foresight equilibrium.

3.2.3 Reservations and Conclusion

As we just saw, borrowing constraints on firms may generate bubbles. Becker, Bosi, Le Van, and Seegmuller (2012) show that similar borrowing constraints applied to heterogenous discounting households within the Ramsey model rule out bubbles. In their model, the lack of bubbles does not mean that the equilibrium is efficient, but an efficient equilibrium rules out bubbles. To summarize, bubbles improve the functionning of firms in production economies whereas bubbles were usually seen as dynamically inefficient before. The recent research has been also considering how governments can secure bubbles, which seems *a priori* curious. We choose to start from the framework of Miao and Wang (2011) and test through this model the different borrowing constraints and assumptions that are necessary to ensure the existence of bubbles, among which, the probability of investment (or stochastic investment function) and the borrowing limits of type Kiyotaki and Moore (1997) and Miao and Wang (2011). In addition we introduce debts with a non-zero interest rate. In chapter 4, we keep the debt structure of Miao and Wang (2011): firms borrow only when the investment opportunity occur. This can be interpretated as *short-term* debts. As proved by chapter 4, prices and capitals of firms are little sensitive to interest rates. In chapter 5 we introduce a new hypothesis of *long-term* debts in the capital of firms.

Chapter 4

Do interest rates on short-term debts impact bubbles?

Contents

The basis framework is the same as Miao and Wang (2011): there is a production economy, and firms have the same production function. The labor adjusts to the demand of firms, to maximize their profits. Firms face stochastic investment opportunities: there is a fixed probability π such that firms can invest at each period. In addition, when firms invest, they can borrow, but debts are restricted to a fraction of the firms' prices. Miao and Wang (2011) show that this framework may generate bubbles in firms' prices. We add a non-zero interest rate on loans to test how this feature may help to control bubbles. Therefore section 4.1.1 presents the same assumptions as the ones of the model of Miao and Wang (2011). They adopt a continuous time formulation, but we rather use a discrete time formulation to get more intuitive results on the debt. This also allows to produce Dynare simulations.

4.1 Layout of the model

4.1.1 Firm: production, capital, wage

We assume that there exist a continuum of firms on $[0, 1]$ with the same , constant return to scale Cobb-Douglas production function. Y_t^m is the production of firm m at time t, K_t^m and N_t^m are respectively the capital and labor of the firm m.

$$Y_t^m = (K_t^m)^\alpha (N_t^m)^{1-\alpha}, \tag{4.1}$$

with $\alpha \in]0, 1[$.

We assume that the labor market is competitive, as Kocherlakota (2009). As a consequence, there is a common wage w_t per unit of labor. We also suppose that this wage is fixed exogenously. We can derive it from solving the usual intertemporal consumption problem in section 4.2.1. We also suppose that the supply of labor perfectly adjusts to the demand of the firms[1].

[1]This assumption is not perfectly neutral on the price of firms, e.g., Letifi and Prigent (2012) analyze how the investment and employment decisions affect the value of firms facing a stochastic demand, depending wether the owner has an option to hire/fire workers, and/or increase/shutdown capital.

Given the wage w_t and the competitive labor market, each firm maximizes at each period the cash-flow of period t by adjusting its level of employment. The firms maximize their benefits:

$$\max_{N_t^m} (K_t^m)^\alpha (N_t^m)^{1-\alpha} - w_t N_t^m. \tag{4.2}$$

This gives the optimal level of employment for firm m:

$$N_t^m = \left(\frac{w_t}{1-\alpha}\right)^{-\frac{1}{\alpha}} K_t^m. \tag{4.3}$$

We can deduce the rate of return on the capital, which is the same for all firms:

$$R_t^m = R_t = \alpha \left(\frac{w_t}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}}. \tag{4.4}$$

The following step is to study the dynamic of the capital of the firms. Usually the evolution of the capital of a firm is subject to the depreciation rate of the capital δ and the periodic investment of the firm. This requires the ability for the firm to realize an investment at each period – each year in our case. As Kiyotaki and Moore (1997), we question this hypothesis: the firms of the economy have different sizes. Precisely we know that the distribution of their sizes follows a Zipf's law, see e.g., (Axtell, 2001) or (Malevergne, Saichev, and Sornette, 2013). There is *a priori* no reason why each firm could realize an investment at each period but on the contrary firms could face difficulties like prohibitive costs or legal barriers. To model this irregular investment behavior, we introduce a probability of investment π as done in Miao and Wang (2011) but also in Kiyotaki and Moore (1997) and Kocherlakota (2009). Such a probabilty should rely on the size of the firms: the biggest firms should be able to invest almost every time whereas the smallest would have little possibilities of investment, according to Kadapakkam, Kumar, and Riddick (1998). For simplicity reasons, we consider this investment probability as a constant. Instead of just a probability of investment, Kunieda and Shibata (2012a) use a random variable, but we do not require such sophistication to get interesting results.

The results of this chapter will be valid only for economies including a large number of middle-sized firms. We will show in the following chapter how this investment probability is a *sine qua non* condition for pricing the equity more than its nominal value.

Suppose that each firm has an opportunity to make an investment at each period with a probability π. If $\pi = 0$, there is no investment possibility, if $\pi = 1$, the firm invests at each period. When the investment is possible, firm m chooses the amount of investment I^m_{t+1}, because K^m_t is the value of the capital at the end of period t. We deduce the equation of variation of the capital:

- $$K^m_{t+1} = (1-\delta)K^m_t \text{ with a probability } (1-\pi), \tag{4.5}$$
- $$K^m_{t+1} = (1-\delta)K^m_t + I^m_{t+1} \text{ with a probability } \pi, \tag{4.6}$$

where δ represents the depreciation rate of the capital. These two equations give:

$$\boxed{K^m_{t+1} = (1-\delta)K^m_t + \pi I^m_{t+1}.} \tag{4.7}$$

We aggregate the firms to get the evolution of the global capital. By equation (4.3) we know that the capital labor ratio is identical for each firm. By aggregating we deduce that:

$$K_t = \int K^m_t dm = N_t \left(\frac{w_t}{1-\alpha}\right)^{\frac{1}{\alpha}} \text{ with } N_t = \int N^m_t dm. \tag{4.8}$$

We put this equality in equation (4.4) and we obtain a global version. Since the rate of return on capital is the same for all firms, it only depends on the aggregate variables.

$$R_t = \alpha K_t^{\alpha-1} N_t^{1-\alpha}. \tag{4.9}$$

We assume that the total labor supply is fixed: $N_t = 1$ and get $R_t = \alpha K_t^{\alpha-1}$. Although the labor market is supposed to be perfectly competitive, we require this strong hypothesis to get results on the global capital.

The aggregate output Y_t can be also simplified:

$$\begin{aligned} Y_t &= \int (K_t^m)^\alpha (N_t^m)^{1-\alpha} dm = \int \left(\frac{K_t^m}{N_t^m}\right)^\alpha N_t^m dm = \left(\frac{K_t}{N_t}\right)^\alpha \int N_t^m dm \\ &= K_t^\alpha N_t^{1-\alpha} = K_t^\alpha. \end{aligned} \quad (4.10)$$

4.1.2 Loans and collateral

We suppose that each firm is owned by one or many risk neutral investors. Each firm is listed on the stock market. The price of the equity logically represents the discounted sum of the future cash-flows. We assume that the investors are risk neutral, because there is no alea on the production functions.

Let $V_t(K_t^m)$ be the stock value at time t of the firm m. The investor-owner wants to maximize the stock value of the firm. The only decision variable of the firm is the amount of the investment – when the opportunity happens. To make this investment, the firm uses the gains of capital of the previous period $R_t K_t^m$ (from equation (4.4)) and takes also one-period loans L_{t+1}^m from the bank with an exogenous interest rate r_t. When there is no possibility of investment, the gains of capital are distributed to the owner(s) of the firm. We suppose that the bank supplies loans perfectly elastically and fixes the interest rate. This is the new feature of the model vis-à-vis Miao and Wang (2011).

There is no possibility for any firm of lending the gains of capital. This also means that when a firm borrows from the bank, the interests of the loan are definitively lost from a social welfare point of view. The model of Miao and Wang (2011) is built with interest-free loans between the households, such that firms not able to invest lend money to the others. This is possible only when the investment probability is low, otherwise it might happen some liquidity problems.

The loans are just "one-period", and the firms only borrow when an investment opportunity happens. As a consequence the capital of firm is purely made of equity, there is no laverage. Since this structure does not fit the reality, we study firms whose capital is composed of equity and debt in the following chapter.

The value of the investment is the only control variable of the investor in the maximization process of a firm's value. It is composed of the gains of capital and the short-term debt:

$$\boxed{0 \leq I_{t+1}^m = R_t K_t^m + L_{t+1}^m. \qquad (4.11)}$$

4.1.3 Choice of the borrowing constraint

There must be a warranty for the bank when the firm borrows, otherwise the firm would never refund the bank. The firm logically pledges a part of its equity. The physical capital K_t^m is "physically" pledgeable, but the stock price of the capital $V_t(K_t^m)$ is much better because it is priced and its value can be compared to the amount of the loan. The bank may not allow the loan to cover the whole capitalization of the firm but a fraction γ of it, for economic, risk or legal reasons.

Two natural constraints are possible, the bank may limit the loan either by $\gamma V_t(K_t^m)$ or by $V_t(\gamma K_t^m)$. These two constraints have been used in the recent litterature. The first one $L_t \leq \gamma V_t(K_t^m)$ has been used by Kiyotaki and Moore (1997). It means that the loan is guaranteed by the liquidation value of a part of the stock value of the firm. The second one $L_t \leq V_t(\gamma K_t^m)$ has been used by Miao and Wang (2011)[2] and they explain that the loan is guaranteed by the value $V_t(\gamma K_t)$ of a small firm whose real capital would be γK_t^m. Indeed if the firm can not refund the loan at the following period, it does not necessarily mean that the investors-owners of that firm are willing to sell a part of their stocks $\gamma V_t(K_t^m)$, they could also choose to change the structure of their firm by selling a part of physical capital γK_t^m. The price of such a capital on the market is $V_t(\gamma K_t^m)$. These two approaches could somehow correspond to some countries where laws protect the workers, or the investors. This is quite difficult to make a choice about the form of the constraint. Shleifer and Vishny (1992) have studied the liquidation values of assets, who appear to be variable and especially depending on the global context of the economy. At first sight it seems that the constraint introduced by Miao and

[2]Compared to the constraint Miao and Wang (2011), there are changes in the time indexes of the capital and in the value to fit the time logic of the model. This detail is explained in next section.

Wang (2011) is a little less realistic that the other one. However the form of their constraint is the key evolution that allow them to find what they call bubbles.

Whatever the constraint's choice, it requires the uniqueness of market prices at each period for all firms and for any level of capital. Indeed the market prices of two firms with the same level of capital is the same because we already assumed that all the firms have the same production function. However, the quotation is not supposed to be linear, so $\gamma V_t(K_t^m) \neq V_t(\gamma K_t^m)$. Indeed, there is *a priori* no proof that the stock price of a firm is proportionnal to its size. On the contrary, the prices of firms may include a size premium: Fama and French (1993) observe that small firms have lower earnings on assets than big firms. This would say that the V_t value function is concave. Obviously, in the other situation, if $\gamma V_t(K_t^m) = V_t(\gamma K_t^m)$, then the two constraints yield the same results.

We choose to use the Miao and Wang constraint for the following. We want to test the response of their model to variations of the bank's interest rate. We will also detail the results obtained with the other because they can be easily related. We name the constraints respectively MW and KM when a comparison is done.

4.1.4 Evolution of the capital of the firms

Once the investment is realized, the firm is supposed to refund the loan with the interests at the following period. To determine the collateral constraint, we consider that the firm pledges a part of the real capital γK_t^m at the beginning of period $t+1$[3], and that the loan is supposed to be refunded with the interests at the beginning of period $t+2$, so the value of the collateral will be $V_{t+1}(\gamma K_t^m)$. The bank delivers the loan L_{t+1}^m at the beginning of period $t+1$ and is asking for $L_{t+1}^m(1+r_{t+1})$ at the following period $t+2$, see Figure 4.1. We can deduce the constraint:

$$\boxed{L_{t+1}^m(1+r_{t+1}) \leq V_{t+1}(\gamma K_t^m), \qquad (4.12)}$$

[3] K_t represents the capital at the end of period t.

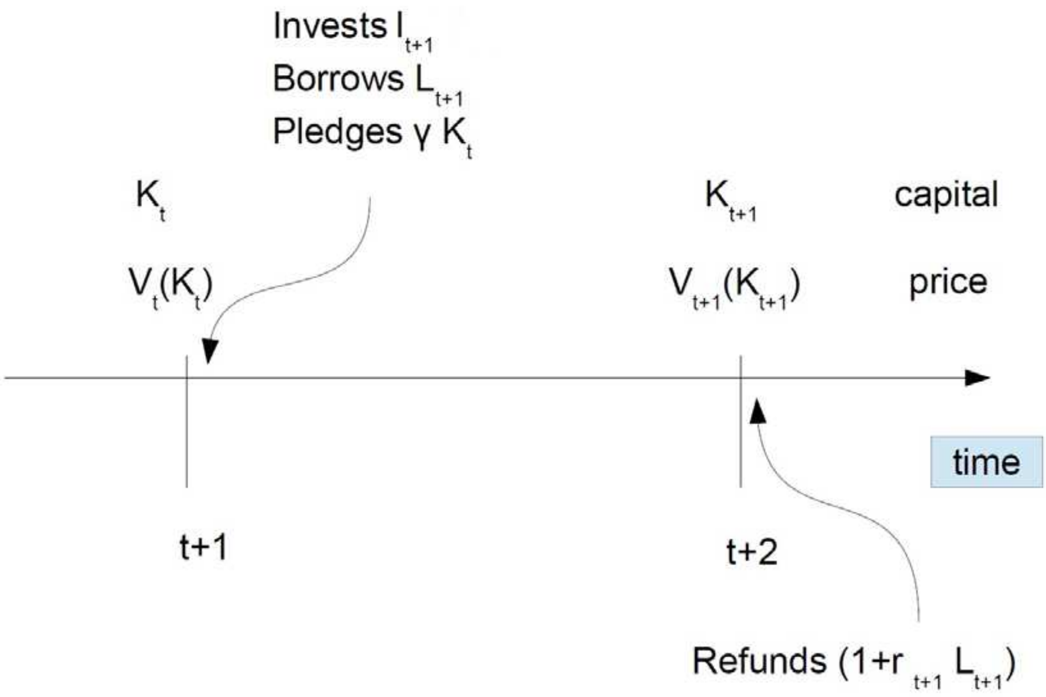

Figure 4.1: Timeline of the investment decision of the firm

which is the same as:

$$L^m_{t+1} \leq \frac{1}{1+r_{t+1}} V_{t+1}(\gamma K^m_t). \tag{4.13}$$

Usually the link between the long-term bank interest rate r_∞ and the discount rate of the investor is: $\beta = \frac{1}{1+r_\infty}$; otherwise the consumer would never save or never consume, by solving its maximization consumption problem (if he would access an asset with yield r_∞). The discount factor of the investors should be related to the growth rate of the economy, but we do not focus on the growth, and we also assume that the long-term interest rate is fixed exogenously by the bank, because it is the limit of the short-term interest rate, also fixed by the bank. The condition $\beta = \frac{1}{1+r_\infty}$ may not be realistic especially if the long-term interest rate is either close to zero or quite high. We have to be careful in the interest rate analysis because changing the long-term interest rate has an influence on the cash-flows, because it

period	capital		cash-flow	
beginning t+1	K_t^m			
investment opportunity	yes	no	yes	no
			$(R_tK_t^m - I_{t+1}^m)^+$	$R_tK_t^m$
beginning t+2	$K_t^m(1-\delta) + I_{t+1}^m$	$K_t^m(1-\delta)$		
			$-(1+r_{t+1})L_t^{m+1}$	0

Table 4.1: Capital variation and cash-flow

represents the interests that a firm pays back but it also changes the value itself, because it changes the discount factor, if the condition holds.

The stock value $V_t(K_t^m)$ of the firm is the discounted sum of the future benefits. To write this valuation correctly we need to consider the dynamics of the capital of the firm. We start from the end of period t. The possibility of borrowing happens at period $t+1$ and is supposed to be refunded at period $t+2$ with the interests. To sum up the values of the capital and the cash-flows, we use Table 4.1, where by definition the debt $L_{t+1}^m = (I_{t+1}^m - R_tK_t^m)^+$ and where $(R_tK_t^m - I_{t+1})^+ = R_tK_t^m - \min(R_tK_t^m, I_{t+1}^m)$ represents the remaining part of the gains of capital if the firm does not borrow at all in period $t+1$.

Using Table 4.1 and the one period discount rate of the owner β, we deduce that the cash-flow of period $t+1$, CF_{t+1} is:

$$CF_{t+1} = (1-\pi)R_tK_t^m + \pi(R_tK_t^m - I_{t+1}^m)^+. \tag{4.14}$$

The refunding of the debt is deduced from cash-flow of period $t+2$:

$$CF_{t+2} = -\pi(I_{t+1}^m - R_tK_t^m)^+(1+r_{t+1}). \tag{4.15}$$

We bring back CF_{t+2} at time $t+1$ by discounting βCF_{t+2} in CF_{t+1}:

$$CF_{t+1} = R_tK_t^m - \pi\left[\min(R_tK_t^m, I_{t+1}^m) + \beta(I_{t+1}^m - R_tK_t^m)^+(1+r_{t+1})\right]. \tag{4.16}$$

The valuation of the capital of firm m is given by the discounted sum of the incoming cash-flows:

$$V_t(K_t^m) = \mathbb{E}_t\Bigg[\sum_{j\geq t}\beta^{j-t}\Big(R_jK_j^m - \pi\Big(\min(R_tK_t^m, I_{t+1}^m) + \beta(I_{t+1}^m - R_tK_t^m)^+(1+r_{t+1})\Big)\Big)\Bigg]. \tag{4.17}$$

We can simplify the previous equation: if the investment exceeds the gains of the capital, $\min(R_tK_t^m, I_{t+1}^m) = R_tK_t^m$ and if the investment is smaller than the gains of capital $L_{t+1}^m = (I_{t+1}^m - R_tK_t^m)^+ = 0$, because there is no debt.

The owner(s) want to maximize the stock value of the firm over the investment decisions. We can write the Bellman equation on the stock value of the firm m:

$$\begin{aligned} V_t(K_t^m) = \max_{I_{t+1}^m} \quad & R_tK_t^m - \pi\left(\min(R_tK_t^m, I_{t+1}^m) + \beta(1+r_{t+1})L_{t+1}^m\right) \\ & +\pi\beta\mathbb{E}_t\left[V_{t+1}\left((1-\delta)K_t^m + I_{t+1}^m\right)\right] \\ & +(1-\pi)\beta\mathbb{E}_t\left[V_{t+1}\left((1-\delta)K_t^m\right)\right]; \end{aligned} \tag{4.18}$$

where $L_{t+1}^m = (I_{t+1}^m - R_tK_t^m)^+$ and the constraint on the debt is given by $L_{t+1}^m(1+r_t) \leq V_{t+1}(\gamma K_t^m)$.

From this equation, two simple situations may emerge. Either the firms never borrows $L_{t+1}^m = 0\ \forall t$, or the firms borrows as much as possible and the constraint is binding $\forall t$: $L_{t+1}^m = \frac{1}{1+r_{t+1}}V_{t+1}(\gamma K_t)$.

When the firm does not borrow, we go back to a classical problem of dynamic programming and from the first order conditions we easily get the form of the value function[4]: $V_t(K_t^m) = v_tK_t^m + b_t$. For this reason, when we study the maximal borrowing possibility, we suppose that the value function has the same special form $V_t(K_t^m) = v_tK_t^m + b_t$, as Miao and Wang (2011) do (guess and verify method). This

[4]A related case is detailed in the following chapter.

value function presents a multiplier term v_t, the shadow price[5], that we will relate to the Tobin's marginal Q, and a shift term b_t that we relate – following Miao and Wang (2011) – to a bubble component of the valuation.

We are especially interested in studying the situation of maximal borrowing because the collateral constraint depends on the valuation of the firm. Borrowing more needs to increase the stock value, maybe until "overevaluation". This simplifies the term $\min(R_t K_t^m, I_{t+1}^m) = R_t K_t^m$ and also the term $(I_{t+1}^m - R_t K_t^m)^+ = L_{t+1}^m = \frac{1}{1+r_{t+1}} V_{t+1}(\gamma K_t^m)$. The investment is given by:

$$\boxed{I_{t+1}^m = R_t K_t^m + \frac{1}{1+r_{t+1}} V_{t+1}(\gamma K_t^m). \qquad (4.19)}$$

This case of maximal borrowing needs the shadow price of the capital to exceed $\frac{1}{\beta}$, because $\frac{1}{\beta}$ is the shadow price of the unconstrained problem: $\boxed{v_t > \frac{1}{\beta}}$. We replace the investment in the Bellman equation we remove the expectations to improve the clarity:

$$\begin{aligned} V_t(K_t^m) =& R_t K_t^m (1-\pi) - \pi \beta V_{t+1}(\gamma K_t^m) \\ &+ \pi \beta V_{t+1} \left((1-\delta) K_t^m + R_t K_t^m + \frac{V_{t+1}(\gamma K_t^m)}{1+r_{t+1}} \right) \\ &+ (1-\pi) \beta V_{t+1} \left((1-\delta) K_t^m \right). \end{aligned} \tag{4.20}$$

Then we replace $V_t(K_t^m)$ by $v_t K_t^m + b_t$:

$$\begin{aligned} v_t K_t^m + b_t =& R_t K_t^m (1-\pi) + \beta b_{t+1} + \pi \beta \frac{v_{t+1} b_{t+1}}{1+r_{t+1}} - \pi \beta b_{t+1} \\ &+ \beta v_{t+1} \left(K_t^m \left(1 - \delta + \pi R_t - \pi \gamma\right) + \frac{\pi}{1+r_{t+1}} v_{t+1} \gamma K_t^m \right) \end{aligned} \tag{4.21}$$

We can derive the value of v_t as well as the value of b_t.

[5]The shadow price represents the net variation on the price of the firm resulting by relaxing one unit of constraint on the investment.

$$\boxed{v_t = R_t(1-\pi) + \beta v_{t+1}\left((1-\delta) + \pi R_t - \pi\gamma + \frac{\pi\gamma v_{t+1}}{1+r_{t+1}}\right),} \tag{4.22}$$

$$\boxed{b_t = \beta b_{t+1} + \pi\beta b_{t+1}\left(\frac{v_{t+1}}{1+r_{t+1}} - 1\right).} \tag{4.23}$$

These are two forward difference equations that give us the behavior of v_t and b_t. We are still missing the values of R_t and K_t. Integrating equation (4.7) on m, because R_t is independant of m, we get the dynamics of K_t:

$$\boxed{K_{t+1} = (1-\delta)K_t + \pi\left(R_t K_t + \frac{\gamma v_{t+1}K_t + b_{t+1}}{1+r_t}\right).} \tag{4.24}$$

Recall the definition of $R_t = \alpha K_t^{\alpha-1}$ and we have the dynamics of all variables: (K_t, R_t, v_t, b_t). Solving the whole system (4.22) (4.23) (4.24) gives us a solution to the initial problem if the variables satisfy the two transversality conditions:

$$\beta^t v_t K_t \quad \to_{t\to\infty} \quad 0; \tag{4.25}$$

$$\beta^t b_t \quad \to_{t\to\infty} \quad 0. \tag{4.26}$$

The term $\pi\beta b_{t+1}\left(\frac{v_{t+1}}{1+r_{t+1}} - 1\right)$ remains strictly positive if $\frac{v_{t+1}}{1+r_{t+1}} > 1$ which requires $v_t > 1 + r_\infty$. We already know that $v_t > \frac{1}{\beta}$ when there is maximal borrowing. If β satisfies the relation $\beta \approx \frac{1}{1+r_\infty}$, then the b component satisfies the transversality condition.

4.2 Results of the model

There are three non-linear equations, we cannot solve the problem directly and we study the stability of the system and try to find the possible steady states, as done by Miao and Wang (2011). Then we study the stability of the system around these points, this allows us to check wether the valuation of the firms may converge.

4.2.1 Global equilibrium

For now, we obtained three equations that govern the firms value. We only focus on the firm's problem to deal with the existence of bubbles. This is *a priori* not the solution to a competitive equilibrium, but indeed it is: we suppose that investors maximize the price of their firms $V_t(K_t^m)$; we suppose that the infinitely-lived households($j \in [0,1]$) maximize their lifetime consumptions c_t^j given wages w_t and prices of the goods p_t, and that they have no access to the bank loans. In addition, we suppose that the households are the investors[6]. We suppose that the bank adjusts exogenously the interest rate[7].

In this case, a competitive equilibrium is the set of global sequences:

$$(Y_t), (C_t), (K_t), (I_t), (N_t), (w_t), (R_t), (p_t),$$

and the sets of individual sequences:

$$(Y_t^m), (C_t^j), (K_t^m), (I_t^m), (N_t^m), (L_t^m),$$

and the set of interest rates r_t such that:

- Investors solve the firm's values maximization problem;
- Households maximize their lifetime consumptions;
- Markets clear: $N_t = 1$ and $C_t = Y_t$ and $p_t C_t = (1-\pi)R_t K_t + w_t - r_t L_t$,

where $(1-\pi)R_t K_t + w_t - r_t L_t$ represents the net income, the rental rate of the capital which is not used for the investment minus the interests paid to the bank. By now, we restrict our analysis to the firm's valuation problem, to check the existence of bubbles and test the effect of the bank in our model.

[6]We could also consider the investors as purely intermediaries, and allow them to consume, by adding another sequence of investors consumptions.

[7]The bank could also set the interest rate to maximize the profits. An example of such maximization of consumers and financers is made in Kunieda and Shibata (2012b).

4.2.2 No bubble

A solution is obtained taking $b_t = 0 \ \forall t$. The valuation of the capital becomes: $V_t(K_t) = v_t K_t$, which eliminates the bubble component. The steady-state values have no t subscript. The long-term interest rate is r. To find them, keeping K and v constant in equation (4.24) we deduce the rental rate of the capital:

$$R = \frac{\delta}{\pi} - \frac{\gamma v}{1 + r_\infty}; \tag{4.27}$$

Then using equation (4.22) we get:

$$v = \frac{\delta(1-\pi)}{\pi\left(1 - \beta(1 - \pi\gamma) + \frac{\gamma(1-\pi)}{1+r_\infty}\right)}. \tag{4.28}$$

Using the previous equation, we deduce the value of R:

$$R = \frac{\delta(1 - \beta + \beta\pi\gamma)}{\pi\left(1 - \beta(1 - \pi\gamma) + \frac{\gamma(1-\pi)}{1+r_\infty}\right)}. \tag{4.29}$$

Given the two last equations, there is a little influence of the bank interest rate on the v steady market value of the capital and on the level of the capital as far as the interest rate remains plausible. However, if we assume that $\beta = \frac{1}{1+r_\infty}$ we get new values of v and R:

$$v = \frac{\delta(1-\pi)}{\pi\left(1 - \frac{1-\gamma}{1+r_\infty}\right)}, \tag{4.30}$$

$$R = \frac{\delta}{\pi}\frac{1 + \frac{\pi\gamma - 1}{1+r_\infty}}{1 + \frac{\gamma-1}{1+r_\infty}}. \tag{4.31}$$

$\gamma\pi < \gamma$ so when the interest rate increases the real rate of return of the capital increases which says that the real capital decreases. The valuation v also decreases if the interest rate increases. This fact seems to match the reality.

We still need to check if $v > \frac{1}{\beta}$. Increasing sharply r_∞ gives $v < 1$ but this is only for exaggerated values of r_∞. In this case the discount rate condition ($\beta = \frac{1}{1+r_\infty}$) would not be credible. This also gives an upper bound on π, since $\lim_{\pi \to 0} v = \infty$ and $\lim_{\pi \to 1} v = 0$.

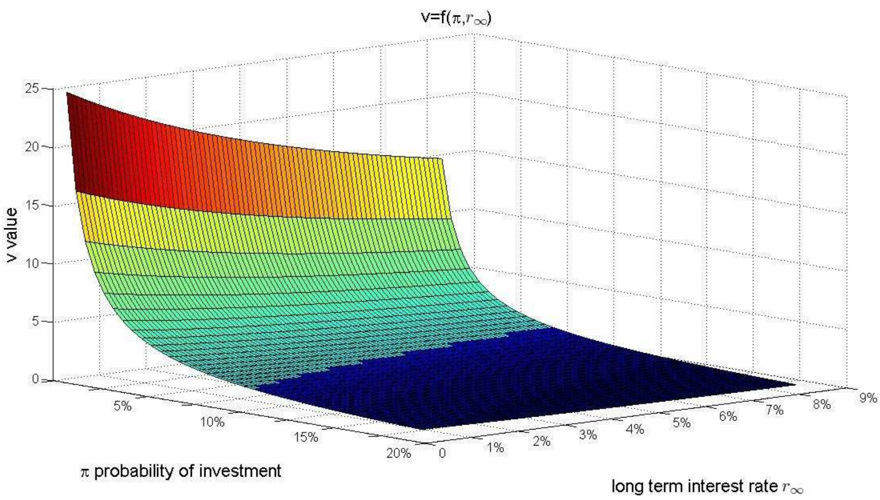

Figure 4.2: Shadow price v of the valuation with respect to π and r_∞

Numerical tests: we take $\delta = 0.025$, $\gamma = 10\%$ and $\beta = \frac{1}{1+r_\infty}$. We calculate the v value. The dark area on Figure 4.2 corresponds to the "impossible" value of v, because within this area $v < \frac{1}{\beta}$ and the maximal investment rule would not be satisfied.

If the discount condition does not hold, v and R still behave the same way, but the curves are smoother. Keeping $\beta = 0.96$ leads to results close to the area of $r_\infty = 4\%$ on the previous figure. When the investment probability decreases, especially under 10%, the v value increases, which says that the capital is highly priced to relax the collateral constraint which is tight. These results are close to the one of Miao and Wang (2011), what we easily understand, looking at the little influence of the bank interest rate. They do not consider that this high v is a bubble, however we could nevertheless interpret this valuation as bubbly. We analyze this vocabulary question at the end of the chapter.

Using our difference equations (4.22) and (4.24) on v and K, we deduce by linarization the behavior of the variables close to the steady state values. The system has exactly one eigenvalue larger than 1 and one eignevalue smaller than 1, as a consequence there is a unique saddle path such that the economy reaches the long run steady state.

As a conclusion, this solution ($b = 0$) exists and is stable. There is a unique path such that the system converges to this equilibrium. In the next section, we study the solutions to the Bellman equation (4.18), such that the bubble component is non-zero.

4.2.3 What about a Miao and Wang bubble?

Suppose that there is a non-zero bubble term b at the steady state. Using the difference equation (4.23) we deduce the new steady state values:

$$v = \frac{(1 + r_\infty)(\pi\beta - \beta + 1)}{\pi\beta}. \tag{4.32}$$

v behaves the same way with respect to π and r_∞ like in the no-bubble equilibrium, though it reaches smaller values. We see that v remains larger than $\frac{1}{\beta}$ as far as the bank interest rate is not too low, because $v > (1 + r_\infty)$ all the time. However, v does not depend on δ and γ, unlike the previous case. The steady state value of the capital K is given by the equilibrium value of the rental rate of capital R by plugging the value of v into its own equation (4.22):

$$R = \frac{(\beta\pi - \beta + 1)(1 + r_\infty)(1 - \beta(1 - \delta) - \gamma(1 - \beta))}{\beta\pi(1 - \pi + (1 + r_\infty)(1 - \beta + \beta\pi))}. \tag{4.33}$$

The value of b is given by the equation on K:

$$b = (1 + r_\infty)K \left(\frac{\delta}{\pi} - R - \frac{\gamma(\pi\beta - \beta + 1)}{\beta\pi} \right). \tag{4.34}$$

R is increasing with respect to the bank interest rate, which is realistic. K decreases with respect to r_∞. As a consequence, the net effect of r_∞ on the bubble remains indefinite. Taking the same values of the parameters as before, the bubble only exists for low values of π: less than 10%. The size of the bubble is decreasing with respect to the bank interest rate, but the trend is very weak.

On Figure 4.3, we draw the bubble value when the discount condition holds: $\beta = \frac{1}{1+r_\infty}$. As we notice, the bubble is eliminated by large values of π or by large

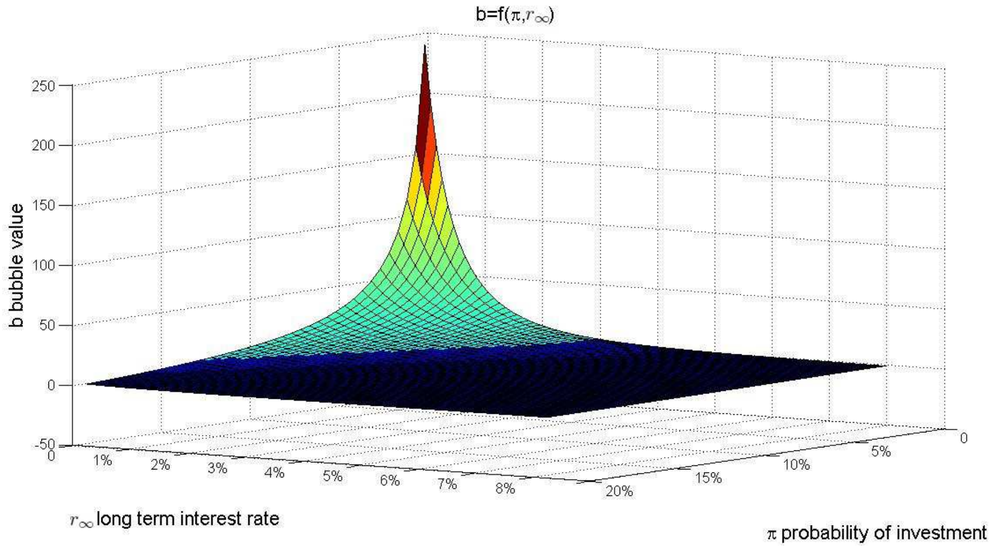

Figure 4.3: Bubble component with respect to π and r_∞

values of long-term interest rate. (The dark area corresponds to a negative or null bubble.)

4.2.4 Stability of the bubble

When the probability of investment is sufficently low and when the long-term interest rate keeps usual values (less than 6%), there exists a steady state of our production economy such that the valuation of the firm presents a consistent "bubble" part – following the definition of Miao and Wang (2011). To know if this valuation can be used to price firms, we need to know if the economy is attracted by this bubbly steady state. To study the stability of this steady state, we linearize the 3 difference equations (4.22), (4.23) and (4.24) around the steady state. We use the notation $G_t = G(1 + \hat{G}_t)$, where $\hat{G}_t$ is the log-linearization of a small deviation from the steady state.

$$\hat{b}_t = \beta\pi \frac{v}{1+r_\infty}\hat{v}_{t+1} + \beta\left(1 + \pi\left(\frac{v}{1+r_\infty} - 1\right)\right)\hat{b}_{t+1}; \tag{4.35}$$

$$\hat{K}_{t+1} = \left((1-\delta) + \pi\alpha R + \pi\frac{\gamma v}{1+r_\infty}\right)\hat{K}_t + \pi\frac{\gamma v}{1+r_\infty}\hat{v}_{t+1} + \pi\frac{b}{K(1+r_\infty)}\hat{b}_{t+1}; \tag{4.36}$$

$$\hat{v}_t = \beta R(\alpha - 1)(\beta\pi - \frac{1}{v})\hat{K}_t + \left(\frac{\beta\pi v}{1 + r_\infty} + 1 - \frac{R}{v}(1 - \pi)\right)\hat{v}_{t+1}. \tag{4.37}$$

Using the steady state values (4.32) and (4.33), the "bubble" equation can be a little simplified:

$$\hat{b}_t = (\beta\pi - \beta + 1)\hat{v}_{t+1} + \hat{b}_{t+1}; \tag{4.38}$$

Let k^* be $\left(1 - \delta + \pi R\alpha + \pi\frac{\gamma v}{1+r_\infty}\right)$. This factor represents the growth rate of K close to the equilibrium, we can distinguish the depreciation rate of the capital $(1 - \delta)$, and the investment, limited by its probablity π, composed of capital gains αR and of debt without the interests $\frac{\gamma v}{1+r_\infty}$. The equation on the capital becomes:

$$\hat{K}_t = \frac{1}{k^*}\hat{K}_{t+1} - \frac{\gamma(\pi\beta - \beta + 1)\gamma}{\beta k^*}\hat{v}_{t+1} - \frac{1}{k^*}\left(\delta - \pi R - \frac{\gamma(\pi\beta - \beta + 1)}{\beta}\right)\hat{b}_{t+1}. \tag{4.39}$$

The whole system can be summurized by the matrix A deduced from the previous equations such that:

$$\begin{pmatrix} \hat{b}_{t+1} \\ \hat{v}_{t+1} \\ \hat{K}_{t+1} \end{pmatrix} = A \begin{pmatrix} \hat{b}_t \\ \hat{v}_t \\ \hat{K}_t \end{pmatrix}. \tag{4.40}$$

By simulation over the range of parameters, the matrix A has three eigenvalues among which two are bigger that 1 and one between 0 and 1. There are two forward looking variables ($\hat{v}_t$, $\hat{b}_t$) and one backward looking variable $\hat{K}_t$, this proves that the system is stable. This means that when the investment rules are restrictive enough, the market valuation of the firm becomes naturally bubbly[8] to satisfy the optimal rule of investment. Moreover, the tranversality condition holds, and therefore this steady state is a solution to the initial problem.

[8]With MW vocabulary...

4.2.5 Market response to changes of debt's limit and investment probability

As explained before, the solution to the Bellman equation depends on the optimal rule of investment. When $v_t > \frac{1}{\beta}$ the shadow price of capital leads to a maximum investment, which in turn generates maximum borrowing. When $v_t \leq \frac{1}{\beta}$ the optimal investment does not require borrowing. The influence of the two parameters π and γ depends on the optimal rule of the firm:

- When $v_t = \frac{1}{\beta}$, the firms are not going to invest as much as possible and not borrow, therefore a change in the investment rules (γ, π) does not change the investment policy[9], the bubble component of the market value of the capital will remain constant to zero and the v-shadow price also keeps constant; (unless the parameters π and γ are such that this is a joint point for the interior solution and the bubbly solution.) As a conclusion a change in the investment rules has no impact on the market value of the firms.

- When the shadow price of capital $v_t > \frac{1}{\beta}$, the bubble component of the market valuation is strictly positive and the firm uses the bubble component to reach the optimal level of investment. Tightening the investment rules (decreasing γ or decreasing π) produces an increase of the bubble b to allow for a higher level of debt.

When the investment rules are restrictive, the valutation of the capital of the firms presents a bubble component. The amount of the loan remains low due to the restrictive collateral constraint. A change in the bank interest rate does not change a lot the results. However, if the long-term discount condition holds: $\beta = \frac{1}{1+r_\infty}$, the steady state values evolve, because this also modifies the discount parameter of the investors, and the prices of the firms. Figure 4.4 gives the evolution of the equilibrium level of the average capital K, its price $vK + b$ and the size of the bubble b, with respect to the long-term interest rate.

[9] A related case is detailed in the following chapter.

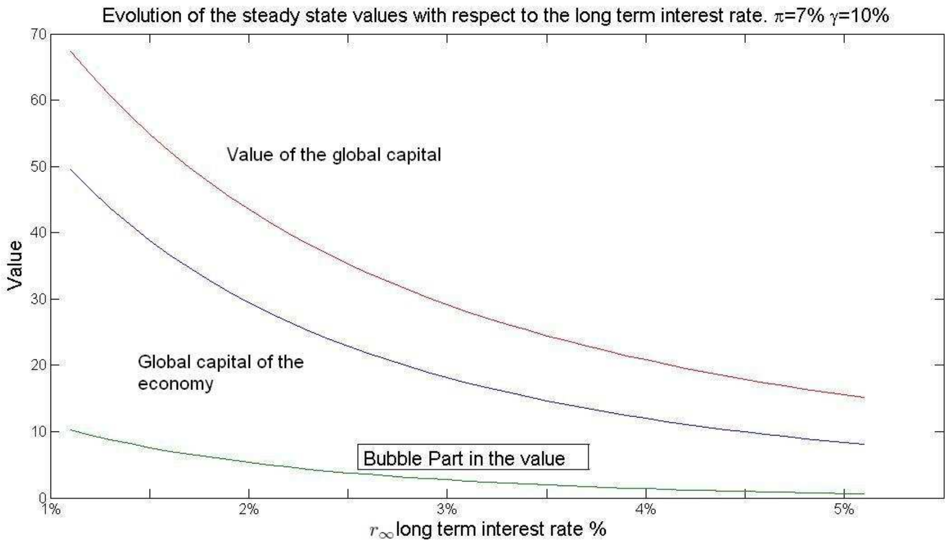

Figure 4.4: Evolution of the equilibrium values: the bubble b, the average capital K and its price $V(K)$ with respect to the long-term interest rate r_∞.

4.3 Shocks on the probability of investment, the collateral limit and the interest rate

We would like to determine the optimal paths of the price's variables v, b and of the capital K, which evolution is given by equations (4.22) (4.23) (4.24), when the investment rules change. This analysis only concerns the situation of maximal borrowing: $v_t > \frac{1}{\beta}$. We consider that the parameters of the model $(\alpha, \beta = \frac{1}{1+r}, \delta, \gamma, \pi, r_\infty)$ are constant and we suppose that the firm parameters (K_t, v_t, b_t) have reached the steady state values that we denote by (K_1, v_1, b_1).

4.3.1 Methodology

To simplify, we assume that there is an **announced, unexpected and permanent modification** of the investment rules. This modification is a change of the parameters π and/or γ. As a consequence the firm will reach in the future the new steady state corresponding to these new investment rules. The average capital K_t will change progressively because it is a stock variable. On the opposite the valuation parameters v_t and b_t will adjust instantaneously to reach an optimal trajectory going to the new steady state, that we denote by (K_2, v_2, b_2).

To find the optimal trajectories, we need to consider the backward and forward terms of the non-linear system. The capital equation is purely backward (4.24) and the valuation terms are forward looking (4.22) and (4.23). The usual solution to this kind of problems should be a mix of backward and forward solutions, as explained by Blanchard (1979a). In our case, to find all the intermediary values, we can numerically determine all the optimal paths going *backward*[10] from the new steady state (K_2, Q_2, B_2) to the previous level of capital K_1. Numerically this needs to invert the non-linear system, add a small deviation from the new steady state (K_2, Q_2, B_2) and calculate all backward paths from this little deviation to the level of capital of the first steady state K_1. We only consider the path(s) joining K_1 because this is the only continuous variable. Since the system is stable we know that there is a unique way to reach the steady state starting from the small deviation.

We still need to know and find exactly which little deviation from the steady state can be considered. Let η be an infinitesimal variation. The variation might be ηK, or ηv, or ηb, or any combination of three of them. Empirically we test all possibilites. We initially exclude all the deviations that lead to irrealistic results: $v < \frac{1}{\beta}$, $b < 0$ and $K < 0$. We also exclude the paths that never reach K_1. We consider all other deviations: the ones whose associated paths reach K_1 and keep positive values of v and b. Numerically we notice that the paths they induce are equivalent, which means that *for an intermediary time they back-converge to the exact same values of* (K_t, v_t, b_t). This outstanding numerical result allows us to conclude on the numerical *uniqueness of the optimal path*. Since all the paths tend to converge, we can consider any deviation among them[11].

Therefore we can calculate the precise numerical trajectories of our productive economy when there is an unexpected and permanent shock in the investment

[10] We can neglegt forward terms because the change of the parameters is unexpected and permanent.

[11] Since the non-linear system is stable, we know that sufficiently close to the steady state, there is a unique path. There is no theoritical warranty when we are far enough from the steady state, but numerically this is true.

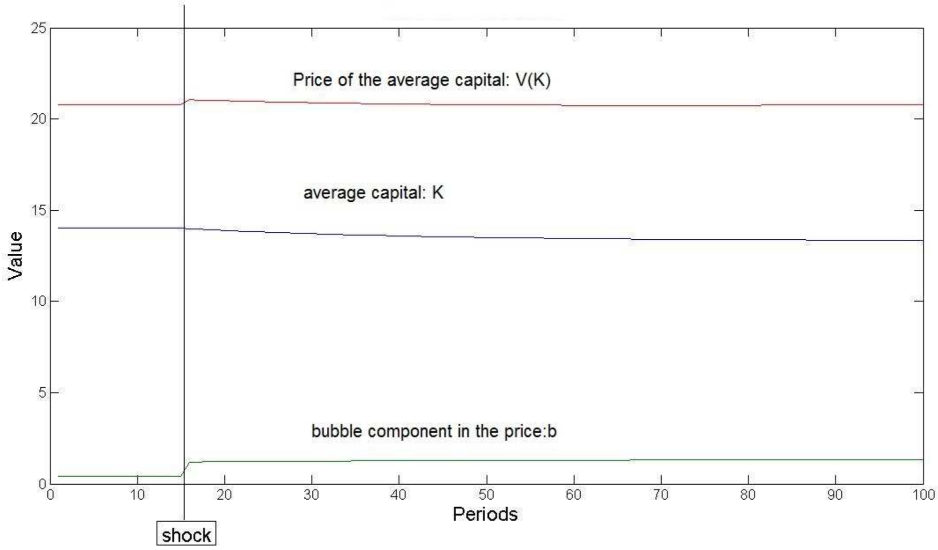

Figure 4.5: Evolution of the equilibrium values for a large negative shock on the collateral limit γ by 50% (credit crunch)

parameters. When the change of the parameters is announced a few periods before, this method still works. If the information about the change(s) is not precise enough, we would have to consider a mix of a backward and a forward solutions, wich could either "smooth" or "emphasize" our following simulations. The interesting case of changing the investment parameters for a fixed number of periods can be treated as two following unexpected changes if the length of the changes is unknown.

Following this way of reasoning, we present two examples of optimal paths for two changes in the investment parameters π and γ. The changes are supposed to be unexpected and permanent.

4.3.2 Restricting the amount of collateral

We calculate the effect on the price of the capital and the evolution of the average capital starting from an initial steady state, and going to the new steady state. The initial parameters values are $\alpha = 0.4, \beta = \frac{1}{1+r_\infty}, \delta = 0.025, r_\infty = 4\%$. The investment probability π remains constant to 10%. The collateral constraint parameter γ skips from 10% to 5%: this represents a large 50% decrease!

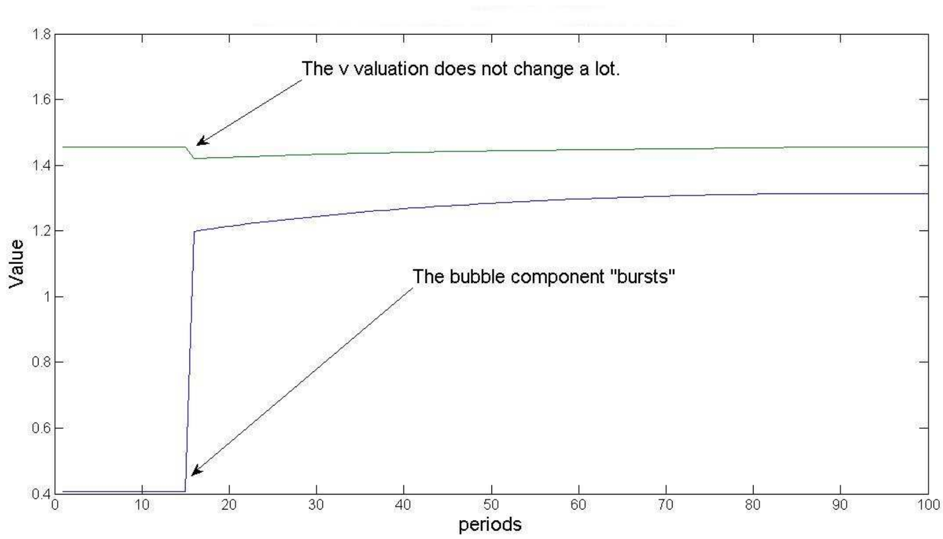

Figure 4.6: Effect on the price components v and b of the negative shock on the collateral limit

On the simulation, Figure 4.5, the average capital slightly decreases after the shock until reaching the new steady state. Reducing the collateral limit decreases the amount of the investment and therefore also decreases the capital. However this effect is very weak. It means that whatever the change of γ in the collateral constraint, it does not change by a large extent the average capital. The v valuation parameter is the same for both steady states because it does not depend on γ, as shown is equation (4.32). On the opposite, the bubble component instantaneously increases. This helps little capitalizations to relax the constraint. The new price of the average capital remains close to the initial one because of the only variation of the b component.

The effect of the variation of the valuation components highly depends on the size of the firms, as shown on Figure 4.6. Two remarks arise from the simulation:

- If we consider a small firm with a small amount of capital, we still have $V_t(K_t^m) = v_t K_t^m + b_t$. This shows that the bubble component is going to create a huge gap on the market price. This underlines how changing the collateral limit for borrowing has a large impact on the valuations of the small capitalizations. On the opposite, when firms have a large amount of

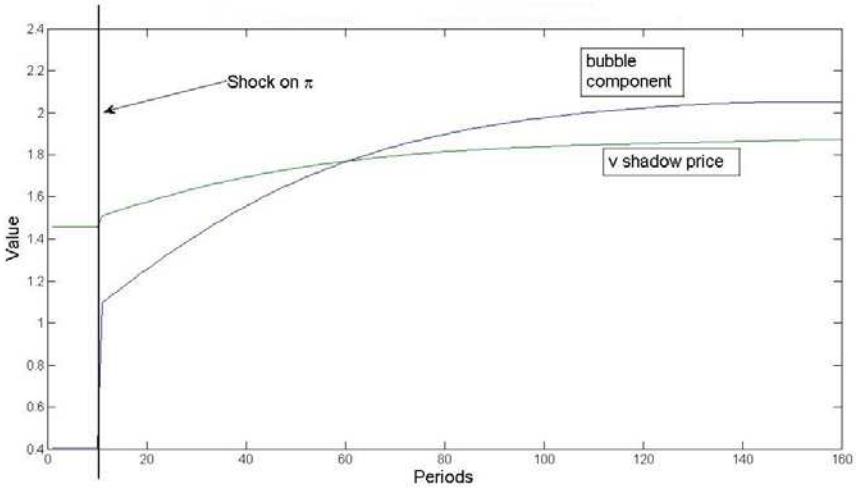

Figure 4.7: Effect on the price components v and b of the negative shock on π by 5%

capital, their market prices are few sensitive to the collateral limit. These conclusions confirm the intuition.

- To get such a sharp increase in b, it is necessary to change γ by 50%, this is hardly believable, unless for a liquidity crisis. In the recent subprime crisis, firms faced credit rationing, (Duchin, Ozbas, and Sensoy, 2010). This can be interpretated as an important shock on γ. However, market prices did not increase to keep the same borrowing level.

4.3.3 Variation of the investment probability π

In the following example Figure 4.7, we suppose that there is a permanent unexpected decrease of the probability of investment π from 10% to 5%. Again we show the evolution of the valuation components.

Decreasing the probability of investment forces the firms to make larger investments when the opportunity occurs. Intuitively, a higher price per unit of capital is the only solution to make larger investments, given the borrowing limits. By simulation, this is what happens, v and b increase. Decreasing the probability of investment penalizes small firms and big firms, while a decrease of γ mainly impact small firms (previous case). Beyond these two examples, it remains impossible to predict exactly the effects of the changes of both parameters. The two components of the prices (v, b) are discontinuous. Their effects might add up and create a jump of the market value or offset and keep the market value smooth.

4.3.4 Do interest rate control bubbles?

The variations of interest rate have very little effect for two reasons. First, on average, firms have low capital because of the investment constraints and the rental rate of capital is high, this easily covers the interest rate of the bank. Second, the amount of the loan is small and the firms pay interests only when the investment opportunity happens. The results of this model can be summarized this way:

Conclusions:

- Reducing the probability of investment at each period leads to an increase in the firms' prices to reach the optimal level of investment. Actually, the literature focusing on prices and investment did not study this question, but rather the sensitivity of investment on firms' prices[a]. For example, Fazzari, Hubbard, and Peterson (1988) show that firms with a high shadow price v are likely to make more investments.
- Reducing the amount of collateral the firms pledge only increase the bubble component of the market value. Credit rationning should generate higher prices. This may be true for growth periods, but false for crises periods. In addition, the collateral limit γ has very little influence on prices, unless shocks are really large.
- Prices increase when the investment becomes sparser.
- Interest rate shocks do not affect prices.

[a]Moreover, the literature is divided on this issue, Kaplan and Zingales (2000)

The“disappointing” conclusions of the model (insignificant effects of the interest rates and the borrowing limit) lead up to think about the significance of this model, and also about the concept of “bubbles”.

4.4 Beyond Miao and Wang bubbles: vocabulary and welfare issues

Extending Miao and Wang (2011) by adding interest rates on loans is not completely satisfactory as we just showed. The authors nonetheless introduced, after determining their equilibria, a probability of switching from the “bubbly” steady state to the other one, as a couple of recent articles do, like for example Kocherlakota (2009) and Kunieda and Shibata (2012a). Needless to say that we could do exactly the same and that would not change the results, except to make the calculations a more complicated... As a conclusion to this model, there could be no central-bank policy exclusively based on interest rates that could help the economy evolve the best way. We adopt a different framework in the following chapter to overhaul the efficiency of interest rates. Before introducing a new framework, we would like to go back to the analysis of what Miao and Wang (2011) call a “bubbly” steady-state.

4.4.1 What is a bubble?

As explained before, we adopted the definition of “bubbly pricing” of Miao and Wang (2011). The additive component b is called a bubble, excluding v, because firms whose capital is really close to zero are valued at least at the bubble price and b is a non-proportionnal part in the valuation. We raise the question wether their definition matches the historical one introduced by Blanchard and Watson (1982). It appears that the non-bubbly equilibrium is just a *linear pricing*, and the bubbly one is an *affine pricing*.

First, the two pricings behave the same way: when $b \neq 0$, v is independant from γ. On the opposite, when $b = 0$, v is varying with respect to π and with respect to γ. We recall here both expressions, v_b and v_{nb} to represent the value of v whether there is a bubble or not. To simplify we take $\beta = \frac{1}{1+r_\infty}$:

- $b \neq 0$:
$$v_b = \frac{(1+r_\infty)(\pi\beta - \beta + 1)}{\pi\beta},$$
- $b = 0$:
$$v_{nb} = \frac{\delta(1-\pi)}{\pi\left(1 - \frac{1-\gamma}{1+r_\infty}\right)}.$$

The no-bubble valuation depends a lot on the γ parameter: the smaller γ, the higher v. Indeed, the "bubbleless" equilibrium pricing v or linear valuation, is increasing when the collateral constraint γ tightens. It is also increasing when the probability of investment π decreases. As a consequence, the linear equilibrium price helps the firms relaxing the investments constraints. To conclude, both pricings (linear and affine) behave the same with respect to the investment constraints.

Let us consider an economy located at the bubbly steady state. We compare the different pricings. The bubbly price is:

$$V_b(K) = vK + b. \tag{4.41}$$

By definition, a firm's fundamental price is the discounted sum of the net cash-flows it generates. From equation 4.18:

$$V_f(K) = \sum_{t\geq 0} \beta^t \left(RK(1-\pi) - \pi(v(\gamma K) + b)\right). \tag{4.42}$$

Numerical application: $\alpha = 0.4$, $\beta = \frac{1}{1+r_\infty} = 0.96$, $\gamma = 5\%$, $\pi = 10\%$. From this, we deduce the equilibrium values: $v = 1.46$, $b = 1.31$ and the equilibrium capital is $K = 13.25$.

The real bubble corresponds to the difference between the two prices $V_b - V_f$. For the equilibrium average value of capital K, both prices are identical, this shows that for the average equilibrium capital, there is no bubble in the price. For low

capitalization	V_b	V_f	$B = V_b - V_f$
1	2.77	-1.60	4.37
K	20.60	20.50	0.1
20	30.43	32.50	-2.07
40	59.56	68.40	-8.84

Table 4.2: Comparison between the bubbly price and the historical one

capitalizations, the fundamental price is lower than the bubbly price, and for higher capitalizations, this is the opposite. To conclude, there is no permanent bubble in the market prices. There are however distortions of prices depending on the capitalizations of firms. The bubbly price helps the little capitalizations.

4.4.2 Welfare of the steady states

It is interesting to compare the steady state values of the two equilibria. We use the same parameters values: $\alpha = 0.4$, $r_\infty = 4\%$, $\beta = \frac{1}{1+r_\infty}$. The consumption is given by: $C = Y - \pi(r_\infty L + I)$ and the debt is: $L = v\gamma K + b$

investment parameters	$\pi = 5\%$ $\gamma = 10\%$	$\pi = 10\%$ $\gamma = 10\%$	$\pi = 10\%$ $\gamma = 5\%$	$\pi = 20\%$ $\gamma = 5\%$
affine price	$v = 1.89$, $R = 0.11$, $K = 9.12$, $b = 1.99$, $C = 2.2$, $P = 19.23$	$v = 1.47$, $R = 0.08$, $K = 14.03$, $b = 0.38$, $C = 2.51$, $P = 21.00$	$v = 1.47$, $R = 0.08$, $K = 14.03$, $b = 1.44$, $C = 2.51$, $P = 22.01$	$v = 1.26$, $R = 0.08$, $K = 16.28$, $b \approx< 0$, $C = 2.56$, $P = 20.5$
linear price	$v = 3.48$, $R = 0.16$, $K = 4.6$, $C = 1.72$, $P = 16.00$	$v = 1.65$, $R = 0.09$, $K = 12.01$, $C = 2.38$, $P = 19.82$	$v = 2.55$, $R = 0.13$, $K = 6.51$, $C = 1.85$, $P = 16.6$	$v = 1.13$ $R = 0.07$, $K = 17.97$, $C = 2.5$, $P = 20.3$

Table 4.3: Comparison between the two steady states

The linear pricing (bubbleless) is not as reliable as the affine pricing: from the consumer (resp. investor, producer) point of view, the consumption (resp. firm average price, capital) is higher when using the affine pricing. In the linear pricing, medium-sized and big firms are favoured, (v is high), and in the affine pricing, the smallest firms are favoured, because of the shift term b. In the affine pricing, small firms are less affected by the investment constraints.

This remark leads us to the constraint problem. The Kiyotaki and Moore (1997) constraint: $L_t < \gamma V_t(K_t)$ applied to the same problem, with the same form of value function $V_t(K_t) = v_t K_t + b_t$ rules out the term b_t. In this situation ($b_t = 0$) both constraints are equivalent:

$$\gamma V_t(K_t) = \gamma v_t K_t = v_t \gamma K_t = V_t(\gamma K_t), \tag{4.43}$$

and they delivers the same results: the solution to the same problem with the KM constraint gives the same results as the linear pricing of the MW constraint.

4.4.3 Interpretation of v in terms of Tobin

Usually the Tobin's Q represents the ratio: value of stock market over the net worth. In the present model this ratio is given by $Q = \frac{V_t(K_t)}{K_t}$. In the linear pricing, $Q = v$, and in the affine pricing Q also depends on K: $Q = v + \frac{b}{K}$. If the global capital is very small, given the term $\frac{b}{K}$ the Tobin's Q reaches high values. The shadow price of the capital v corresponds to the marginal q. The Tobin's q (on Figure 4.8) has been criticized especially because it was failing to predict investment correctly, for example in Blanchard, Rhee, and Summers (1993). In this model, the Tobin's q reflects the investment given the investment constraints: it increases when investment opportunities and collateral constraints tighten. When all constraints are relaxed, and the firms do not reach the maximal investment, $Q = v = \frac{1}{\beta}$. When the market crashes, usually the stock values collapse, the firms reduce investment, and the regulators tend to make the investment easier to improve the state of the economy, which precisely in our model, decreases v. So maybe the Tobin's q reflects better the investment taking into account the investment restrictions.

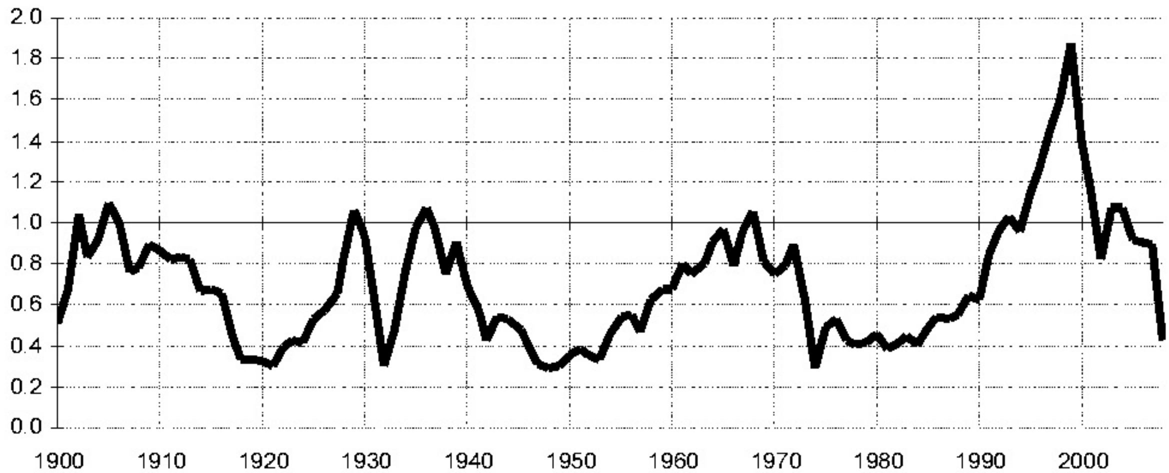

Figure 4.8: Tobin's Q ratio 1900-2009[a]

[a]Source: The Federal Reserve, Blanchard et al. (1993)

4.5 Conclusion of the model

To reach the maximal investment i.e. $v > \frac{1}{\beta}$, the probability of investment π and the collateral amount γ have to be very low and this seems unrealistic. If we intended to limit borrowing of firms by 50% of their values, it would require that the probability of investment π is less than 0.001. This means that each firm would invest on average every 1000 years, this never happens actually. These restrictions on parameters hardly limit the level of debt.

We also disagree about the ability of the model of Miao and Wang (2011) to create a "bubble". We acknowledge the interest of the constraint they introduce, which creates two different pricings while the usual borrowing constraint does not.

Unfortunately, interest rates have little effect (almost zero) on the prices of the firms, as far as the discount rate of the investor does not depend on it. Indeed, the interest rate reduces the remaining part of the production which is available for the consumption but this is the only effect: the aggregate consumption is given by $C_t = Y_t - \pi(I_t - r_t L_t)$. We were expecting some influence on the valuations. Though the model shows intuitive and nice results to changes in investment policy, the sensitivity of variables such as prices and capital remains low.

Given the low credibility of the numerical results, and the weakness of the model to reproduce the effect of the bank interest rate on prices, we introduce in the next chapter a more realistic capital structure of firms, to include a permanent part of debt (long-term debt). We do not change the other hypothesis.

Chapter 5

Long-term debts and bubbles [1]

Contents

[1]Presented at the AFSE 2013 conference.

In the previous chapter, we considered that the capital of the firms was excusively composed of equity, and firms were borrowing short-term debts when they were investing. The debt was designed to help the firms increase the level of investment when the opportunity of investment happened. Unfortunately, interest rates on debts have little impact, because amounts of the debt are low, and debts only exist when the investment opportunities occur. We also proved that there exist no bubble in firms' prices, but two different pricings, one affine and one linear pricing. The existence of these two pricings depend on the form of the borrowing constraint, the one of Kiyotaki and Moore (1997) or the one of Miao and Wang (2011). The present chapter changes the financial structure of firms to include a permanent part of long-term debt in the capital. In this case, the capital is composed of debt and equity. Instead of relaxing the borrowing constraint, the debt is a production factor. As a consequence, this model requires the existence of a debt market.

With this new structure of firms, we show that interest rates have a significant effect on firms, precisely they influence the amounts of capital and the prices. In addition, we are interested to test the robustness of this new capital structure to the form of the borrowing constraint. Apart from this new feature, the following model keeps the same assumptions as the previous chapter.

This new capital structure emphasizes the uniqueness of the way of pricing firms, and this pricing does not depend on the choice of the constraint. Prices are quite sensitive to interest rates. Generally, prices are free from bubbles. Nevertheless, we exhibit a particular relation between the parameters of the model (limit of borrowing, probability of investment), such that prices are bubbles.

The first part exposes the model, the second establishes the Bellman equation and the difference equations. Sections 3 and 4 analyze the pricing systems and their responses to interest rate variations, and last section concludes.

5.1 Structure of the capital

The assumptions about the economic context of this model correspond exactly to the ones of Chapter 4 section 1. We quickly recall them and then we detail the new features of the model.

5.1.1 General informations

We assume that there exists a continuum of firms with the same "Cobb Douglas" production function, with constant return to scale. Y_t^m is the production of firm m at time t, K_t^m and N_t^m are respectively the capital and labor of the firm m.

$$Y_t^m = (K_t^m)^\alpha (N_t^m)^{1-\alpha}, \tag{5.1}$$

with $\alpha \in]0,1[$. The aggregate output is Y_t. There is a unique wage w_t per unit of time. There is a fixed supply of labor, $N_t = 1$. Firms optimize their benefits at each period with respect to their employment. As a consequence, the rental rate of capital is the same for all firms: $R_t = \alpha Y_t^{1-\alpha}$ and the aggregate output satisfies the same condition as before: $Y_t = K_t^\alpha N_t^{1-\alpha}$. The depreciation rate of the capital is δ and there is a probability π such that each firm can make an investment at each period:

- $K_{t+1}^m = (1-\delta)K_t^m$ with a probability $(1-\pi)$; (5.2)
- $K_{t+1}^m = (1-\delta)K_t^m + I_t^m$ with a probability π. (5.3)

The new structure of the capital of the firms is composed of equity and debt. For a firm m, the equity is denoted by E_t^m and the debt by D_t^m. We have $K_t^m = E_t^m + D_t^m$. The debt is supplied perfectly elastically by the bank, which also fixes the interest rate. The firms only borrows from the bank and have no access to capital gains of other firms, which are paid to investors and consumed.

The equity of firms is priced on the stock market: the value of firm m at time t is: $V_t(E_t^m)$.

5.1.2 Borrowing limit

The bank limits the debt of each firm by a fraction of the value of the firm: to borrow, the firm pledges a part of his equity. This is designed to guarantee that the firm's debt is sustainable: the firm has to pay the interests of his debt, and the firm has to be able to repay the whole debt. The amount the firm pledges is such that the firm has no interest to lose the equity instead of refunding the bank. The collateral constraint obviously concerns the equity E_t^m instead of the whole capital K_t^m, to avoid securing the debt by another debt. All along this chapter we consider the two constraints that are used in the recent litterature.

- the KM (Kiyotaki and Moore, 1997) constraint:

$$D_t^m \leq \gamma V_t(E_t^m), \tag{5.4}$$

- the MW (Miao and Wang, 2011) constraint:

$$D_t^m \leq V_t(\gamma E_t^m). \tag{5.5}$$

The investment relies on the gains of the capital $R_t K_t^m$. At each period, the firm pays the interest on the debt, $r_t D_t$. Since the capital depreciates, the amount of the equity changes at each period and the amount of debt might do the same.

We suppose that the rental rate of capital is higher than the bank interest rate, otherwise the firm would never borrow from the bank: $R_t > r_t$. The financial cash-flow at each period is the yield on whole capital minus the interests of the debt:

$$\begin{aligned} R_t K_t^m - r_t D_t^m &= R_t(E_t^m + D_t^m) - r_t D_t^m, & (5.6)\\ &= R_t E_t^m + (R_t - r_t) D_t^m. & (5.7) \end{aligned}$$

While $R_t - r_t > 0$ capital gains on the debt part are strictly positive, because the objective function equation (5.7) is linear in D_t^m, and R_t does not depend on firm

m:

$$\max_{D_t^m} R_t K_t^m - r_t D_t^m. \tag{5.8}$$

As soon as $R_t - r_t > 0$, the firm borrows as much as possible. This remains true, whatever the choice of the constraint. When $R_t - r_t > 0$ the collateral constraint (MW or KM) is binding.

At each period the whole capital depreciates, K_t^m becomes $K_t^m(1-\delta)$. The capital is composed of equity and debt, therefore the sum $E_t^m + D_t^m$ depreciates. As we just proved, at each time, the borrowing constraint must be binding. For the sequel of the chapter, we adopt KM constraint: at any time t the debt of the firm must satisfy $D_t^m = \gamma V_t^m(E_t^m)$. At time $t+1$ the capital of the firm satisfies the same relation: $D_{t+1}^m = \gamma V_{t+1}^m(E_{t+1}^m)$. The values of E_{t+1} and D_{t+1}^m depend on the possibility of investment. In addition, if there is a variation of the debt: $D_{t+1}^m \neq D_t^m$, there must be a transfer to the bank. This variation $(D_{t+1}^m - D_t^m)$ can be included either in the cash flow of the period, or from the next period capital.

Normally, when a firm wants to refund a bank, it uses the capital gains. If the variation of the debt $(D_{t+1}^m - D_t^m)$ is included in the cash flow, we get the following equation on the investment:

$$\begin{aligned} 0 \leq I_t^m &\leq R_t K_t^m - r_t D_t^m + D_{t+1}^m - D_t^m \\ &\leq R_t(E_t^m + \gamma V_t^m(E_t^m)) - r_t \gamma V_t^m(E_t^m) + \gamma V_{t+1}^m(E_{t+1}^m) - \gamma V_t^m(E_t^m). \end{aligned} \tag{5.9}$$

The investment I_t^m is the control variable of the firm. To solve the problem, the control variable must not depend on the value of next period state variable. However, equation (5.9) shows that I_{t+1}^m is correlated to E_{t+1}^m by term $\gamma V_{t+1}^m(E_{t+1}^m)$. In addition $\frac{\partial}{\partial E_{t+1}^m} V_{t+1}(E_{t+1}^m) > 0$, because increasing I_{t+1}^m increases E_{t+1}^m, which in turns increases the maximal bound of the investment I_{t+1}^m. This allows for a Ponzi scheme. Technically, this means that the correspondance is not compact.

Because of this technical point, we must include the variation of the debt in the next period capital K_{t+1}^m. The value of the debt at time $t+1$ is different wether the investment happens $D_i(t+1)$ or not $D_n(t+1)$:

- $K^m_{t+1} = (1-\delta)K^m_t + D^m_{n(t+1)} - D^m_t$ with a probability $(1-\pi)$, (5.10)
- $K^m_{t+1} = (1-\delta)K^m_t + I^m_t + D^m_{i(t+1)} - D^m_t$ with a probability π. (5.11)

These two equations are analog to (4.5) and (4.6) of the previous chapter. The investment is bounded from above by the financial cash-flow of the period:

$$0 \leq I^m_t \leq R_t K^m_t - r_t D^m_t = R_t E^m_t + (R_t - r_t) D^m_t. \tag{5.12}$$

We can write the Bellman equation. We are pricing the equity E^m_t instead of the whole capital.

$$\begin{aligned} V_t(E^m_t) = \max_{I^m_t \text{ satisfying eq. (5.12)}} \quad & R_t E^m_t + (R_t - r_t) D^m_t - \pi I^m_t \\ + \quad & \pi \beta V_{t+1}(E^m_{i(t+1)}) \\ + \quad & (1-\pi)\beta V_{t+1}(E^m_{n(t+1)}); \end{aligned} \tag{5.13}$$

where $E^m_{i(t+1)}$ and $E^m_{n(t+1)}$ are the respective values of the equity whether the investment occurs or not. $R_t E^m_t + (R_t - r_t) D^m_t - \pi I^m_t$ represents the cash-flow of period $t+1$ minus the investment. We determine the next-period values of the equity:

$$E^m_{n(t+1)} + D^m_{n(t+1)} = K^m_{n(t+1)} = (1-\delta)K^m_t + D^m_{n(t+1)} - D^m_t. \tag{5.14}$$

This gives:

$$E^m_{n(t+1)} = E^m_t(1-\delta) - \delta D^m_t. \tag{5.15}$$

The same way, we get the equation on $E^m_{i(t+1)}$:

$$E^m_{i(t+1)} = E^m_t(1-\delta) - \delta D^m_t + I^m_t. \tag{5.16}$$

This leads to the complete Bellman equation of the price of the equity:

$$\begin{aligned} V_t(E_t^m) = \max_{0 \le I_t^m \le R_t E_t^m + (R_t - r_t) D_t^m} & \quad R_t E_t^m + (R_t - r_t) D_t^m - \pi I_t^m \\ & + \quad \pi \beta V_{t+1}(E_t^m (1-\delta) - \delta D_t^m + I_t^m) \\ & + \quad (1-\pi) \beta V_{t+1}(E_t^m (1-\delta) - \delta D_t^m). \end{aligned} \tag{5.17}$$

The transversality condition is:

$$\beta^t V_t(E_t^m)^t \rightarrow_{t \to \infty} 0. \tag{5.18}$$

To solve this equation, we distinguish two main cases, when the solution is interior, and when the investment constraint is binding.

5.2 Solving the Bellman equation

5.2.1 Interior solution

Interior solution means that the constraint of the control variable will never be binding: $I_{t+1} < R_t E_t^m + (R_t - r_t) D_t^m$ and the first order conditions are satisfied. Even if it does not create any bubble, it remains very useful to get an intuition on the form of the value function V. We adopt the Kiyotaki and Moore constraint: $D_t^m \le \gamma V_t(E_t^m)$. As explained before this is an equality $D_t^m = \gamma V_t(E_t^m)$ because the borrowing constraint is binding (5.8). The MW case gives the same results. The Bellman equation becomes:

$$\begin{aligned} V_t(E_t^m) = & \max_{0 \le I_t^m \le R_t E_t^m + (R_t - r_t)\gamma V_t(E_t^m)} \\ & R_t E_t^m + (R_t - r_t) \gamma V_t(E_t^m) - \pi I_t^m \\ & + \pi \beta V_{t+1}(E_t^m (1-\delta) - \delta \gamma V_t(E_t^m) + I_t^m) \\ & + (1-\pi) \beta V_{t+1}(E_t^m (1-\delta) - \delta \gamma V_t(E_t^m)). \end{aligned} \tag{5.19}$$

The first order condition with respect to the investment:

$$V'_{t+1}(E_t^m(1-\delta) - \delta\gamma V_t(E_t^m) + I_t^m) = \frac{1}{\beta}; \tag{5.20}$$

where $V'(x)$ is the first order derivative of V. The FOC with respect to the state variable E_t^m gives:

$$\begin{aligned} V_t'(E_t^m) =& R_t + (R_t - r_t)\gamma V_t'(E_t^m) \\ & + \pi\beta V'_{t+1}\left(E_t^m(1-\delta) - \delta\gamma V_t(E_t^m) + I_t^m\right)\left(1-\delta-\delta\gamma V_t'(E_t^m)\right) \\ & + (1-\pi)\beta V'_{t+1}\left(E_t^m(1-\delta) - \delta\gamma V_t(E_t^m)\right)\left(1-\delta-\delta\gamma V_t'(E_t^m)\right). \end{aligned} \tag{5.21}$$

Substitute (5.20) in (5.21) and we get:

$$\boxed{R_t + (R_t - r_t)\frac{\gamma}{\beta} = \frac{1-\beta}{\beta} + \delta + \frac{\delta\gamma}{\beta}.} \tag{5.22}$$

We remark that the rental rate of the capital does not depend on π but on the borrowing limit γ. Actually this is logical: the investment constraint is assumed not to be binding, which means that the firm can reach the optimal level of investment. The probability of investment does not affect the average level of capital, and the rental rate of the capital. We understand that R_t has to cover the subjective interest rate of the investor: $\frac{1-\beta}{\beta}$ and the depreciation rate of capital δ. The rental rate excess with respect to the debt $(R_t - r_t)$ associated to its proportion in the capital γ has to cover the depreciation of the capital financed by the debt $\delta\gamma$. The rental rate of capital can be also expressed as:

$$R_t = \frac{1-\beta+\delta\beta+\delta\gamma+\gamma r_t}{\beta+\gamma}. \tag{5.23}$$

For all values of $\gamma \in [0,1]$, R_t remains larger than r_t, the borrowing constraint keeps binding. When r_t increases, R_t also increases and the capital reduces, as well as the debt. Again, we need to be careful with long-term interpretations, depending on the validity of the discount condition $\beta = \frac{1}{1+r_\infty}$, where r_∞ is the limit of the short-term bank interest rate r_t. The derivative of R_t with respect to the borrowing parameter γ is:

$$\frac{\partial R_t}{\partial \gamma} = \frac{-1+\beta+\beta r_t}{(\beta+\gamma)^2}. \tag{5.24}$$

The positivity of the numerator depend on the link between β and r_t: $\frac{\partial R_t}{\partial \gamma} > 0 \Leftrightarrow \beta > \frac{1}{1+r_t}$. When the subjective discount rate of the agents r_c such that $\beta = \frac{1}{1+r_c}$ is lower than the bank interest rate $r_c < r_t$, investors reduce the whole capital when the borrowing constraint is relaxed, they consume more. This is the opposite as the usual behavior of a household which stores a part of his wealth in a bank asset. Indeed, when the debt is more costly, the investors decrease the capital to pay less interests.

Increasing the bank interest rate reduces the capital of the firm, it reduces the investment, and the debt. Relaxing the debt limit (increasing γ) increases the level of capital if the bank interest rate is sufficiently low: $r_t < \frac{1-\beta}{\beta}$.

We deduce from (5.20) that the valuation of the firms is given by:

$$V_t(E_t^m) = \frac{E_t^m}{\beta} + b_t; \tag{5.25}$$

where b_t is a common constant that depends on the period. To find b_t we substitue equation (5.25) in the Bellman equation (5.19):

$$b_t(1-\gamma(R_t-r_t)+\delta\gamma) = \beta b_{t+1} \tag{5.26}$$

We use the equation (5.23) to get:

$$\boxed{b_t\frac{1+\gamma+\gamma r_t}{\beta+\gamma} = b_{t+1}} \tag{5.27}$$

Because[2] $1+\gamma r_t > \beta$ the solution to equation (5.27) diverges. It can still be a solution to the Bellman equation, only if it satisfies the transversality condition, equation (5.18).

$$\beta^t V_t(A_t^m) = \beta^{t-1} A_t^m + \beta^t b_t. \tag{5.28}$$

[2] We assume $r_t > 0$.

The first term converges to zero, and the second is:

$$\begin{aligned}\beta^t b_t &= \beta^t b_0 \prod_{t=0}^{t-1} \frac{1+\gamma+\gamma r_t}{\beta+\gamma} \\ &= b_0 \prod_{t=0}^{t-1} \frac{\beta+\beta\gamma+\beta\gamma r_t}{\beta+\gamma}. \end{aligned} \tag{5.29}$$

To know if equation(5.29) converges to zero, we must study the limit of the term $\frac{\beta+\beta\gamma+\beta\gamma r_t}{\beta+\gamma}$. Ler r_∞ be the limit of the short-term interest rate:

$$\frac{\beta+\beta\gamma+\beta\gamma r_\infty}{\beta+\gamma} < 1 \iff \beta < \frac{1}{1+r_\infty}. \tag{5.30}$$

Let r_c be the discount rate of the consumers: $\beta = \frac{1}{1+r_c}$. In this case, equation (5.29) converges to zero if and only if:

$$\boxed{r_c > r_\infty.} \tag{5.31}$$

If equation(5.31) is true, the transversality condition (5.18) is true and any price of type:

$$V_t(E_t^m) = \frac{E_t^m}{\beta} + b_0 \prod_{t=0}^{t-1} \frac{1+\gamma+\gamma r_t}{\beta+\gamma} \tag{5.32}$$

is a solution to the Bellman equation (5.19). In addition this solution has a diverging bubble component in the price. To know if this solution exists, we must study the behavior of the bank.

In section 5.1.2, we proved that the firm was borrowing if and only if the rental rate of the capital R_t is larger than the bank interest rate r_t. This condition when $t \to \infty$ becomes:

$$\frac{1-\beta+\delta\beta+\delta\gamma+\gamma r_\infty}{\beta+\gamma} > r_\infty. \tag{5.33}$$

In this equation (5.33), we can substitute $\beta = \frac{1}{1+r_c}$. We get a new condition on r_c:

$$\boxed{r_c+\delta+\delta\gamma > r_\infty,} \tag{5.34}$$

which is compatible with the previous condition, equation (5.31). The remaining question is to determine the behavior of the bank.

5.2.2 The bank optimizes

To rule out bubbles, the bank can set the long-term interest rate to $r_\infty > r_c$. However, if the bank optimizes the profits, the value of r_∞ is the solution to the bank's problem:

The bank maximizes the limit of the sequence of one-period problems by solving:

$$\max_{r_\infty} r_\infty D_t, \tag{5.35}$$

with D_t such that $K = E_t + D_t$ and R is given by equation (5.23): $R = \frac{1-\beta+\delta\beta+\delta\gamma+\gamma r_\infty}{\beta+\gamma}$. To link E_t and D_t, we consider equation (5.32). To simplify let us write $B_t = B_0 \prod_{t=0}^{t-1} \frac{1+\gamma+\gamma r_t}{\beta+\gamma}$. Because R is constant, we know that K is constant. At each time t we have:

$$D_t = \gamma V_t(E_t^m) = \gamma \frac{E_t^m}{\beta} + B_t. \tag{5.36}$$

Because $V_t(E_t^m)$ diverges if $E_t^m \geq 0$, to keep $D_t + E_t$ constant, we need to consider a negative equity:

$$\begin{aligned} & K = \gamma \left(\frac{E_t^m}{\beta} + B_t \right) + E_t \\ \iff \quad & E_t = (K - \gamma B_t) \frac{\beta}{\beta + \gamma}. \end{aligned} \tag{5.37}$$

The bank problem at time t becomes:

$$\begin{aligned} & \max_{r_\infty} r_\infty \gamma V_t \left((K - \gamma B_t) \frac{\beta}{\beta+\gamma} \right) \\ = \quad & \max_{r_\infty} r_\infty \frac{K}{\beta} + \frac{\beta B_t}{\beta+\gamma} \\ = \quad & \max_{r_\infty} r_\infty \frac{1}{\beta} \frac{R}{\alpha}^{\frac{1}{\alpha-1}} + \frac{\beta}{\beta+\gamma} B_0 \prod_{t=0}^{t-1} \frac{1+\gamma+\gamma r_t}{\beta+\gamma}. \end{aligned} \tag{5.38}$$

The profit of the bank is strictly increasing in r_t, which states that the bank should fix r_∞ as high as possible. However by equation (5.31), the firm borrows if and only if $r_c > r_\infty$. There is no well defined solution to this problem, because the domain of r_∞ is open. Any value of r_∞ strictly lower but closed to r_c guarantees to the bank a large and increasing profit.

There is a state of the economy, such that the price of the equity includes a non-zero growing bubble. There is a Ponzi sheme on the debt due to the bubble, which still satisfies the transversality condition. This state generates a very unsual situation of negative equity for the firm. If we impose a reasonable condition of positive equity, this rules out the bubble in a finite time T, when $E_T^m = 0$.

If we suppose that the firm has a long-term positive equity, the bubbly term is zero: $B_0 = 0$. The price of the firm is therefore $V_t(E_t) = \frac{E_t}{\beta}$. We can deduce the value of the debt: $D_t = \gamma V_t(E_t) = \frac{\gamma E_t}{\beta}$. The value of the equity comes from the value of R in equation (5.23):

$$K = \frac{R^{\frac{1}{\alpha-1}}}{\alpha} = D_t \left(1 + \frac{\gamma}{\beta}\right) \tag{5.39}$$

We deduce that the debt is constant over time, and the value of the debt is:

$$D = \frac{\beta}{\beta+\gamma} \left(\frac{1-\beta+\delta\beta+\delta\gamma+\gamma r_\infty}{\alpha(\beta+\gamma)}\right)^{\frac{1}{\alpha-1}}. \tag{5.40}$$

The optimization function of the bank is:

$$\max_{r_\infty} \left[r_\infty \frac{\beta}{\beta+\gamma} \left(\frac{1-\beta+\delta\beta+\delta\gamma+\gamma r_\infty}{\alpha(\beta+\gamma)}\right)^{\frac{1}{\alpha-1}} \right]. \tag{5.41}$$

This gives the optimal long term interest rate of the bank, expressed with the discount rate of the investor r_c:

$$\boxed{r_\infty = (r_c + \delta\beta + \delta\gamma)\frac{1-\gamma}{1-\gamma+\gamma\alpha}. \tag{5.42}}$$

With the value of r_∞, we can deduce the value of R, and the complete problem (investor, bank, consumer) has a unique solution.

- If the equity of the firms is positive, the KM constraint rules out the hypothetic constant term. As a consequence the valuation does not depend on the time, and becomes:
$$V(E_t^m) = \frac{E_t^m}{\beta}. \tag{5.43}$$
The bank optimizes its profits by setting the long term interest rate. There is a unique solution to the model.

- If the firms could be in negative equity, there would exist a bubble in the prices of the firms. This bubble would be geometrically growing over time, and would create a Ponzi scheme on the debt. This would also allow for unlimited profits for the bank, while the long-term interest rate on debts would remain smaller than the subjective discount rate of the investor r_c.

We proved in this part that the investment probability π is "useless" when the investment constraint is not binding, i.e. when the shadow price of capital does not exceed $\frac{1}{\beta}$. We also remark the formulation of the value function in (5.25) that we will impose to find the boundary solution. If we adopt the Miao and Wang point of view: the constraint becomes $V_t(\gamma K_t^m)$, the results are the same because the derivative of $V_t(\gamma K_t^m)$ also gives a factor γ.

5.2.3 Maximal investment

In this part, we consider intially the Miao and Wang constraint[3]. We go back to the Bellman equation and now we suppose that the shadow price of the capital exceeds $\frac{1}{\beta}$: when the investment opportunity happens, the firm is willing to invest as much as possible, and the investment reaches its maximal level: $I_t^m = R_t E_t^m + (R_t - r_t) V_t(\gamma E_t^m)$, because $D_t^m = \gamma V_t(E_t^m)$. We replace this expression in the Bellman equation and we also substitute the value function by $V_t(E_t^m) = v_t E_t^m + b_t$.

[3]The KM case is done in Appendix C.

We get the following equation:

$$\begin{aligned} v_t E_t^m + b_t = & \left(R_t E_t^m + (R_t - r_t)(\gamma v_t E_t^m + b_t)\right)(1-\pi) \\ & + \beta v_{t+1}(E_t^m(1-\delta) - \delta(\gamma v_t E_t^m + b_t)) + \beta b_{t+1} \\ & + \pi \beta v_{t+1}(R_t E_t^m + (R_t - r_t)(\gamma v_t E_t^m + b_t)). \end{aligned} \tag{5.44}$$

We identify the terms that depend on E_t^m and the others to get two difference equations on v_t and b_t:

$$\boxed{v_t = \frac{\beta v_{t+1}\left(\pi R_t + 1 - \delta\right) + R_t(1-\pi)}{1 - \gamma(R_t - r_t)(1-\pi) + \beta\gamma v_{t+1}\left(\delta - \pi(R_t - r_t)\right)};} \tag{5.45}$$

$$\boxed{b_t\left(1 - ((R_t - r_t)(1-\pi) + (R_t - r_t)\pi\beta v_{t+1} - \delta\beta v_{t+1})\right) = \beta b_{t+1}.} \tag{5.46}$$

We still need to consider the equation on the global capital K_t to get the complete dynamic system. We know that for each firm we have: $K_t^m = E_t^m + V_t(\gamma E_t^m)$ because the investment constraint is binding, so we need to get the evolution of E_t^m:

$$E_{t+1}^m = (1-\delta)E_t^m - \delta V_t(\gamma E_t^m) + \pi\left(R_t E_t^m + (R_t - r_t)V_t(\gamma E_t^m)\right). \tag{5.47}$$

We aggregate all the firms to get:

$$\boxed{E_{t+1} = (1-\delta)E_t - \delta(v_t\gamma E_t + b_t) + \pi\left(R_t E_t + (R_t - r_t)(v_t\gamma E_t + b_t)\right).} \tag{5.48}$$

Since $K_t^m = E_t^m + V_t(\gamma E_t^m) = E_t^m(1+\gamma v_t) + b_t$ we deduce the equation on the average capital:

$$\begin{aligned} \frac{K_{t+1} - b_{t+1}}{1+\gamma v_{t+1}} = & (1-\delta)\frac{K_t - b_t}{1+\gamma v_t} - \delta\left(v_t\gamma\frac{K_t - b_t}{1+\gamma v_t} + b_t\right) \\ & + \pi\left(R_t\frac{K_t - b_t}{1+\gamma v_t} + (R_t - r_t)\left(v_t\gamma\frac{K_t - b_t}{1+\gamma v_t} + b_t\right)\right). \end{aligned} \tag{5.49}$$

Instead of considering this "not so easy to use" equation, we focus on the equity equation (5.48) and make the link between E_t and R_t:

$$\boxed{R_t = \alpha K_t^{\alpha-1} = \alpha \left(E_t(1+\gamma v_t) + b_t\right)^{\alpha-1}. \qquad (5.50)}$$

The whole system is now given by equations (5.45) and (5.46), (5.48) and (5.50). We impose the transversality conditions to the solutions.

$$\begin{aligned} \beta^t v_t E_t &\to_{t\to\infty} 0, \\ \beta^t b_t &\to_{t\to\infty} 0. \end{aligned} \tag{5.51}$$

Given the complete system, we look for the equilibrium values of the variables. This allows to find the price(s), and the equilibrium global level(s) of capital of the production economy.

5.3 Linear pricing

5.3.1 Equilibrium values of the model

Taking $b = 0$ is a solution to equation (5.46). The solution of $b = 0$ is the same for both constraints (MW and KM) for linearity reasons. We use the two equations (5.45) and (5.48) and get the following results. The variables without t subscript denote the steady-state values. We obtain the following system:

$$v = \frac{R(1-\pi)}{1-\beta-\gamma(R-r_\infty)(1-\pi)},$$

$$v = \frac{\delta - \pi R}{\gamma(\pi(R-r_\infty)-\delta)}.$$

Together they give the values of R and v:

$$R = \frac{\delta(1-\beta) + \delta\gamma r_\infty(1-\pi)}{\pi(1-\beta)}, \tag{5.52}$$

$$v = \frac{\delta\gamma(1-\pi)}{\gamma\left(\pi(1-\beta) - \delta\gamma(1-\pi)\right)}. \tag{5.53}$$

The rental rate of capital R is inversely proportionnal to the probability of investment π. This shows that when investment opportunities rarefy, the rental rate of the capital increases, and the capital decreases. This corresponds to smaller firms with higher rate of return on the capital. When the collateral constraint γ is relaxed, the rental rate of the capital slightly increases. This corresponds to the interests on the depreciation of the debt $\propto \delta r_\infty$.

The rental rate of capital depends on the debt interest rate of the bank, it increases when the bank interest rate increases. The effect of the bank interest rate is emphasized by the scarcity of investment.

Curiously the v shadow price of the capital does not depend on the bank interest rate. On Figure 5.1, we draw v as a function of π and γ to know when $v > \frac{1}{\beta}$, necessary condition for maximal investment. On this graph and the following ones, we limit the values to 20, though it reaches higher values. The same way, the lower limit is 0 even if the values are negative. The dark blue area corresponds to the values of v such that $v < \frac{1}{\beta}$, where the solution does not exist.

On Figure 5.1, we remark a curve along which v diverges. Looking at equation (5.53), we understand that when $\pi \rightarrow_{>} \frac{\delta\gamma}{1-\beta+\delta\gamma}$, $v \rightarrow \infty$. We will consider how interpretating this phenomenon in the following. Above this curve, for any fixed value of γ, v is decreasing with respect to π which means that there exists a threshold such that $v = \frac{1}{\beta}$ and we reach the interior solution. In this model, the combinations of π and γ that allow $v > \frac{1}{\beta}$ do not especially correspond to restrictions in investment or collateral but on the opposite, the values are quite large. To check the validity of the borrowing constraint, we check that R remains larger than the long-term bank interest rate $r_\infty = 4\%$ for any couple (π, γ).

We can draw the whole domain where the steady state of the linear pricing exists.

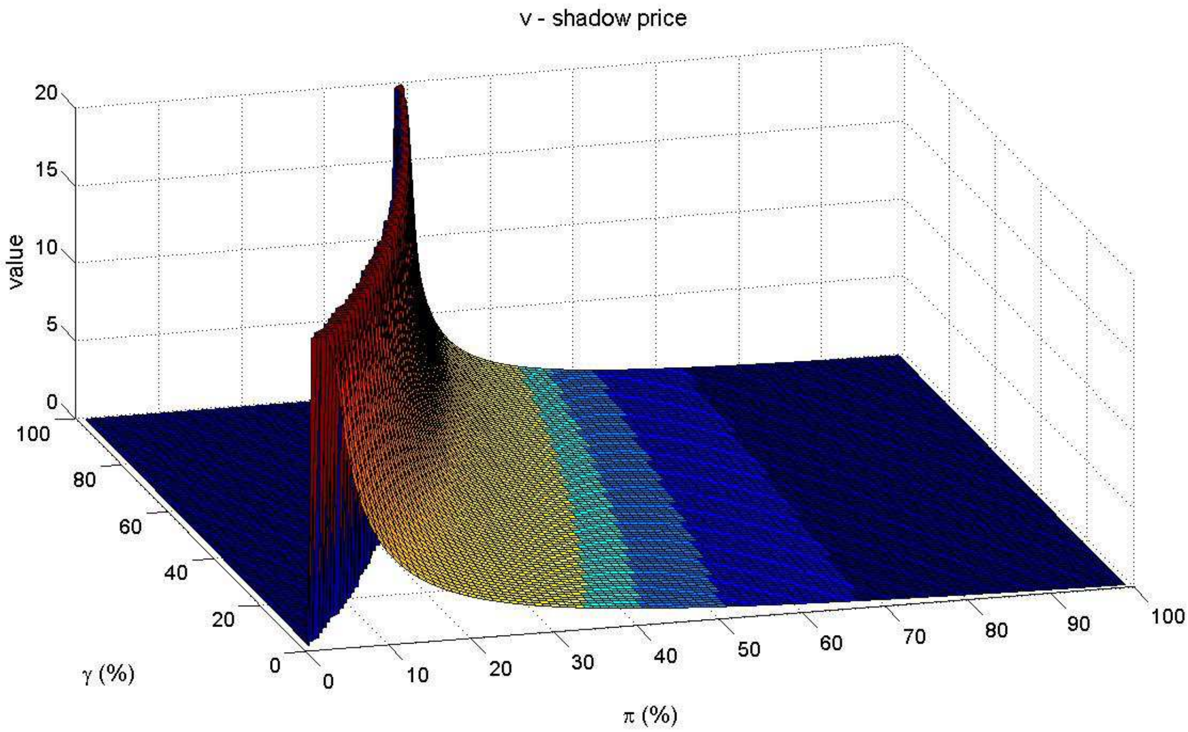

Figure 5.1: Shadow price v in maximal investment with respect to π and γ

In the blue zone, only the interior solution exists, $v = \frac{1}{\beta}$. On the π upper limit of the red zone[4], $v = \frac{1}{\beta}$ and the value of R of the linear pricing correspond to the value of R of the interior solution. This means that when π increases, the equilibrium steady state of the linear pricing becomes *continuously* the steady state of the interior solution.

On the left border of the red zone, R behaves continuously while v diverges. This means that $v\gamma$ diverges and $K = E + D$ remains smooth. However $E \geq 0$ and $D \geq 0$. The equity of the firm $A \to 0$ and the shadow price v diverges to keep the product $v\gamma A$ smooth. On this limit the equity converges to zero and the capital is only composed of debt. Along this curve, there exists a bubble. The value of the debt is $v\gamma E = D$ and the value of the capital is $K = E + D = 0 + D$. There is a bubble, and the price of the bubble is $\frac{K}{\gamma}$.

[4]Right border of the red part Figure 5.2.

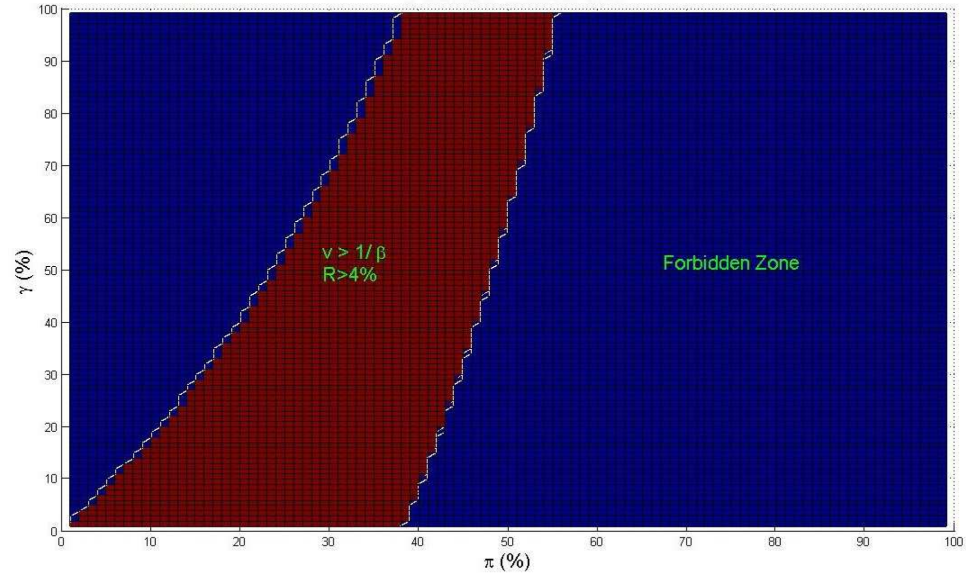

Figure 5.2: Existence domain of the shadow price v: $v > \frac{1}{\beta}$ and $R > r_\infty = 4\%$

The performance of this model relies on the ability of the economy and the prices to reach a steady state for quite large and realistic values of π and γ, in which the capital is priced over its "usual" value: $v > \frac{1}{\beta}$. There also exists a relation between the investment parameters: $\pi = \frac{\delta\gamma}{1-\beta+\delta\gamma}$ such that when it is satisfied, the capital of firms is only composed of debts, the amount of the equity is zero, but the equity has a net positive price, there is a real bubble in the economy. The value of the bubble is $\frac{K}{\gamma}$.

By linearization and numerical tests, we prove that the steady state is stable where it exists.

5.3.2 The bank's equilibrium interest rate

Again, to get a global equilibrium, let us consider that the bank optimizes its profits by setting the value of the debt interest rate. At each period, the global amount of the debt is $v\gamma E$. The value of the capital is $E + D = E(1 + v\gamma)$. We deduce E from $R = \alpha K^{\alpha-1} = \alpha E^{\alpha-1}(1 + v\gamma)^{\alpha-1}$. The bank problem can be stated this way:

$$\max_{r_\infty \text{such that eq.(5.55)}} r_\infty \gamma \frac{\delta\gamma(1-\pi)}{\gamma(\pi(1-\beta) - \delta\gamma(1-\pi))} E. \qquad (5.54)$$

$$\frac{\delta(1-\beta)+\delta\gamma r_\infty(1-\pi)}{\pi(1-\beta)} = \alpha E^{\alpha-1}\left(1+\gamma\frac{\delta\gamma(1-\pi)}{\gamma(\pi(1-\beta)-\delta\gamma(1-\pi))}\right)^{\alpha-1} \quad (5.55)$$

When E is fixed, increasing r_∞ increases the bank's profit. Looking at the constraint, an increase of r_∞ leads to an increase of $E^{\alpha-1}$. But $(\alpha-1)<0$, therefore increasing r decreases E. To conclude there is a unique solution to the bank problem, which determines the level of capital of the firms.

5.3.3 Interest rate shocks

In this section, we focus on the variations of prices and of capital when the capital is priced over $\frac{1}{\beta}$. This correspond to the part "linear pricing".

We evaluate the impact of an unexpected shock on the bank's interest rate on the price of the firm. We take $\alpha = 0.4$, $\beta = 0.96$, $r_\infty = 4\%$, $\gamma = 90\%$, $\pi = 40\%$. *The interest rate mean is* 4% *and is shocked by* 0.5% *(absolute value).* The steady state values are: $v = 6$, $R = 9.6\%$, $E = 1.68$. The price of the average equity is $V(E) = 10.07$. On the following Figures 5.3, 5.4 and 5.5 these steady-state values correspond to the red lines. We expose first the effect of a non-persistent shock, lasting *1 period*, on Figure 5.3.

The unexpected shock on the interest rate decreases the equity, but increases the shadow price v_t. The average price of firms decreases by less than 0.1%, because the effect on v_t is slightly delayed compared to the one on the equity E_t. The rental rate of the capital logically increases, which means that the global capital first decreases, to refund the increase of the interests and then progressively goes back to its initial level.

The results are *amplified if the shock is more persistent.* Suppose that the interest rate shock is decreasing (AR-1 process, with autoregression coefficient $\rho = 0.4$) lasting 5 years, on Figure 5.4. The responses of the shadow price v_t and the equity E_t are amplified, but on average, the price of the equity $V_t(E_t)$ is not changed compared to the shock lasting 1 period.

If the amplitude of the shock is increased, simulations show that the response of the equity and the price are also proportionnaly increased.

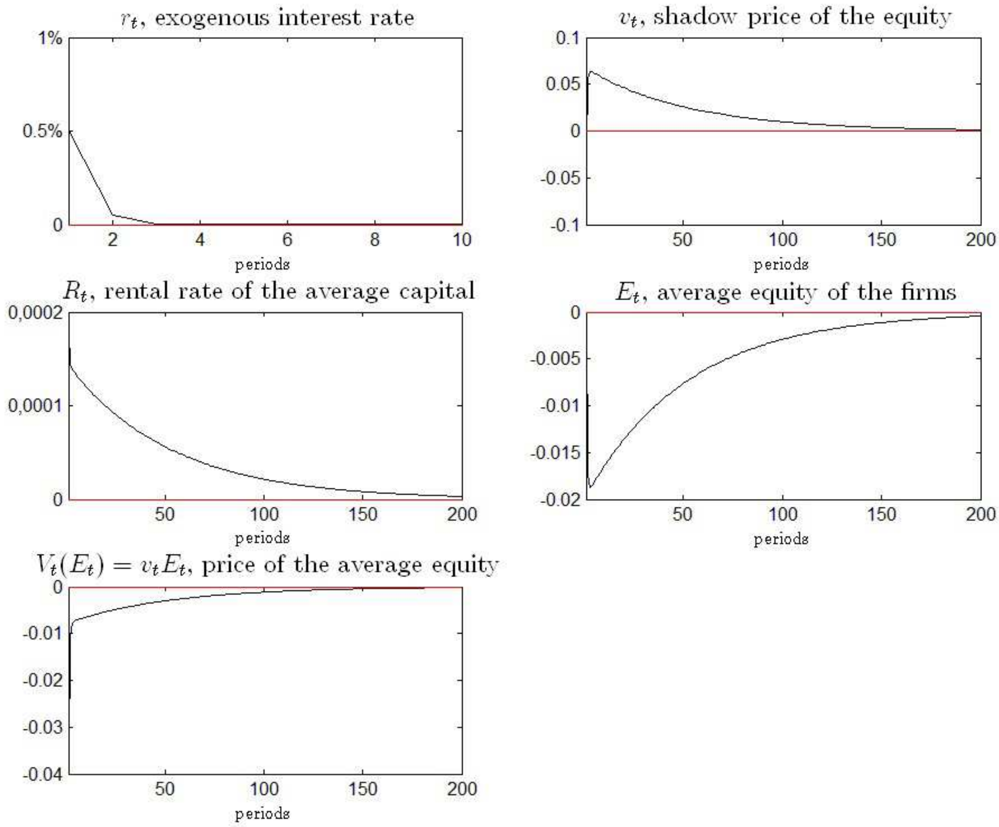

Figure 5.3: Evolution of the equilibrium values in response to a shock of 0.5% on the interest rate lasting 1 year

However, the price of the equity is highly impacted is the shock is lasting over more periods[5]. For example, on Figure 5.5, the shock is lasting 20 years ($\rho = 0.9$ in the AR-1 process).

The main difference, due to the persistence of the shock, is the *initial and persistent decrease of the price $V_t(E_t)$ by more than* 2%, which leads to think that there is a *fall of* 2% *on the markets*. Obviously, if the amplitude of the shock on the interest rate was higher, this would generate a larger fall.

[5] We cannot study the effects of large interest and persistent interest rate shocks because we reach Dynare's limits.

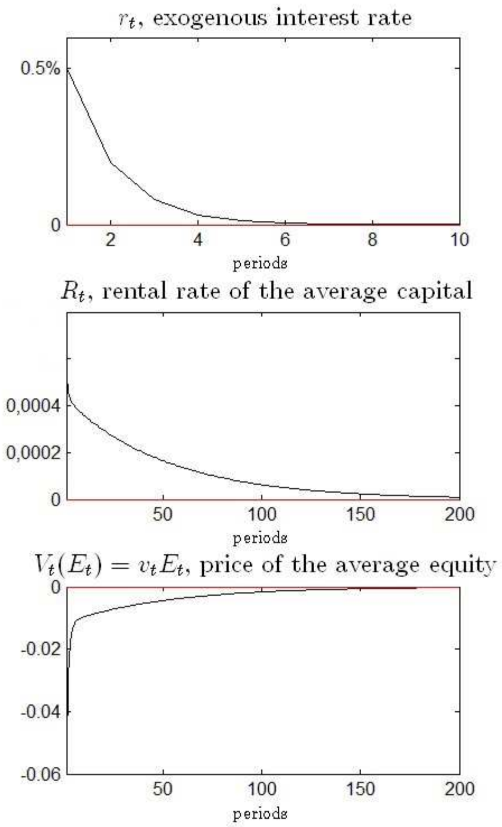

Figure 5.4: Evolution of the equilibrium values in response to a shock of 0.5% on the interest rate lasting 5 years

Prices of firms are very responsive to interest rate shocks. The shadow price increases with a positive shock on the bank interest rate. It tends to counterbalance the decrease of the net equity. On average, market prices decrease by a small extent. The return to the equilibrium (prices and equity) is very long compared to the length of the shock. When the interest rate shock is more persistent, the shadow price first jump down at the shock, like the equity. This can be considered as “stock market fall”.

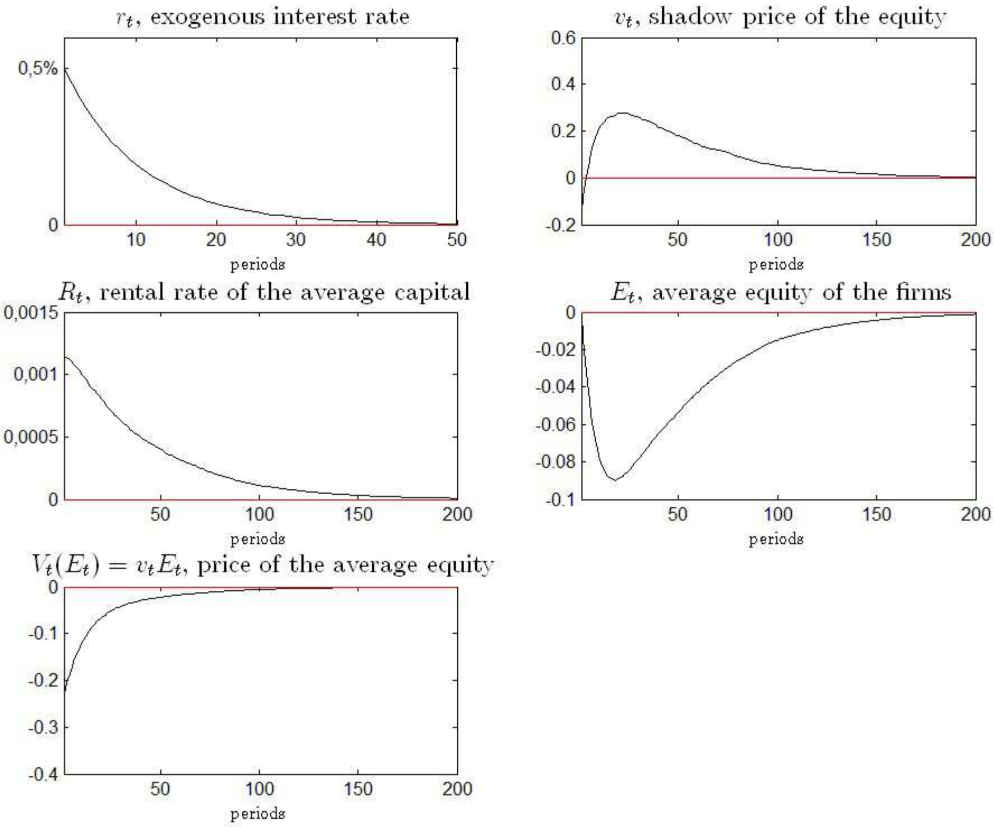

Figure 5.5: Evolution of the equilibrium values in response to a shock of 0.5% on the interest rate lasting 20 years

5.3.4 Shocks on investment

We expose first the results of a 5% (absolute) shock on the limit of borrowing γ lasting 5 years. We adopt the same parameters' values as in the section 5.3.3 presenting shocks on the interest rate.

The effects are unpredictable. Indeed, if γ increases, we are waiting for a decrease of v_t to adjust the net borrowing $v_t \gamma E_t$. Actually *the pricing v_t increases and the equity reduces*. This increase of 5% on γ creates a 2% increase of v_t and decreases E_t by 0.4%. Though it might be difficult to understand the response of the economy, we remark that the price is more volatile than the equity. The rental rate of the capital does not change a lot, which means that the capital is almost

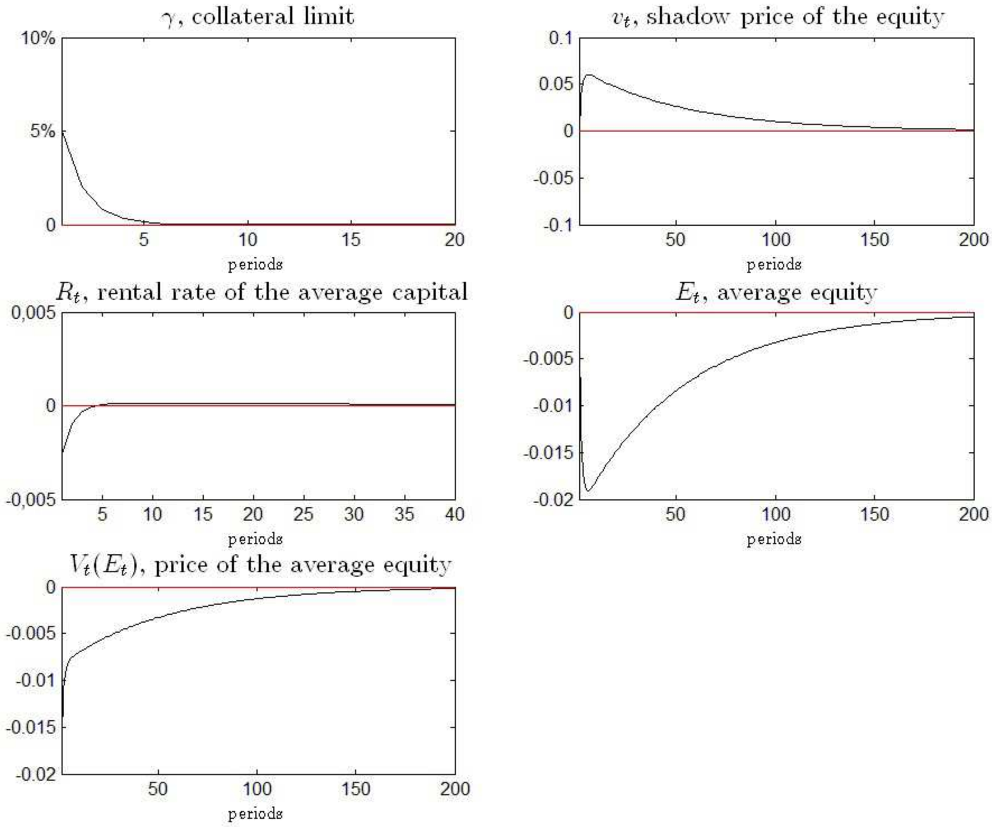

Figure 5.6: Evolution of the equilibrium values in response to a shock of 5% on the borrowing limit γ

constant. The equity reduces and the debt increases for the shock. To conclude about the behavior of this linear pricing, we simulate the effect of a shock of 5% (relative shock of 11%) on the probability of investment π.

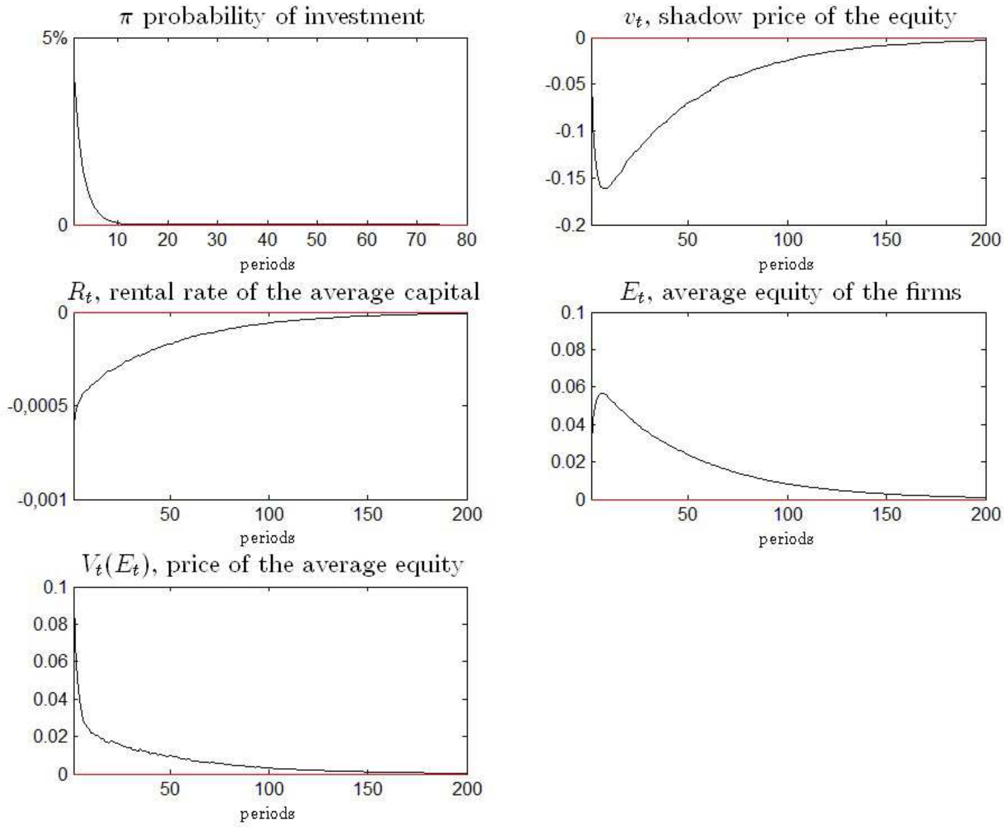

Figure 5.7: Evolution of the equilibrium values in response to a shock of 5% on the investment probability π

The 5% shock on π generates an increase of the equity. In addition, even if the shadow price decreases, there is an increase of the price of the equity, which means that the debt increases for the shock. The rental rate of the average capital decreases, which shows that the capital increases. A positive shock on the probability on investment means that it is easier to invest. As a consequence, the valuation of the firm can be relaxed a little bit, because the firm is able to invest more often. Even if the shadow price of the equity reduces, a positive shock on π generates a global increase of market prices.

To conclude about the linear pricing, the volatility of the price and the equity is identical, both are highly impacted by interest rate variations. A positive shock on the interest rate reduces the equity, and reduces a little the whole capital. The shadow price increases, except if the shock is persistent, in this case, it decreases first. Relaxing the borrowing limit increases the shadow price and reduces the average equity, and increasing the probability of investment increases the equity and decreases the shadow price. In all cases, the effect on the equity exceeds the effect on the shadow price, as a consequence, market prices evolve like the equity, but “smoother”.

5.4 Multiple equilibria

We are also interested by the existence of an affine pricing as solution to the maximal investment case, and we would like to see how it depends on the borrowing constraint.

The equilibrium values of the affine pricing exist, see Appendix D. However the system is unstable and the bubble component does not satisfy the transversality condition. This affine pricing does not exist either when using the KM constraint, see Appendix C.

We analyzed all interior solutions, and all situations of maximal investment with affine prices. Only three solutions coexist:

- The shadow price of the capital v is exactly $\frac{1}{\beta}$ and the average capital of firms is determined by R through equation (5.23). Since $K = E + D = E + v\gamma E$, we can deduce all equilibrium values.

- The shadow price of the capital v strictly exceed $\frac{1}{\beta}$, the firms invest as much as possible. The value of v is given by equation (5.53) and the value of R by equation (5.52).

- The parameters satisfy $\pi = \frac{\delta\gamma}{1-\beta+\delta\gamma}$ and the value of the equity is a pure bubble, the real capital is only composed of debt.

Since all these equilibrium values are stable, there is no reason to choose one or the others. From a welfare point of view, the first case has lower values of R and therefore a higher capital. From an investor point of view, the bubble allows to generate income with a zero equity. The solution may depend on the endowments at the initial time of the problem.

In a production economy, where the capital of firms is composed of equity and debt, limitation of the collateral and stochastic investment opportunities bring firms to invest a lot when possible. There exists just one equilibrium pricing excluding any form of bubble. It also does not depend on the type of constraint (MW or KM). The shadow price and the equity are both volatile, but negatively correlated. Consequently, market prices' variations are small. However, market prices might be discontinuous when interest rate shocks are persistent. When the interest rate of the bank increases, the average equity decreases because of the cost of the interests, and the prices increase to keep the debt laverage constant, unless the shock is persistent and the prices crash. Any increase in the interest rate reduces the amount of global capital and therefore the welfare. Increasing the probability of investment increases the equity and the market prices. However, this probability of investment must remain lower than 1 to guarantee the existence of a solution with maximal investment, and the linear pricing. When the borrowing limit is relaxed, the equity decreases but the global capital and the shadow price increase. The debt takes the place of the equity which therefore keeps a high yield.

Appendix C

Boundary solution with the KM constraint

We apply the same reasoning as in section 5.2.3, starting from the Bellman equation at the end of section 2, equation (5.19). The equation on v_t does not change, but the one on b_t becomes:

$$b_t \left[1 - \gamma \left((R_t - r_t)(1-\pi) + \left((R_t - r_t)\pi - \delta\right) \beta v_{t+1}\right)\right] = \beta b_{t+1}. \tag{C.1}$$

We remark that the domain of transversality is the same as the one resulting from the MW constraint. The equation on the average equity as well as the equation on the rental rate of the capital evolve:

$$E_{t+1} = (1-\delta)E_t - \delta\gamma(v_t E_t + b_t) + \pi \left(R_t E_t + (R_t - r_t)\gamma(v_t E_t + b_t)\right), \tag{C.2}$$

$$R_t = \alpha K_t^{\alpha-1} = \alpha \left(E_t(1+\gamma v_t) + \gamma b_t\right)^{\alpha-1}. \tag{C.3}$$

The steady state value change because the equation on the bubble leads to:

$$v = \frac{1 - \beta - \gamma(R - r_\infty)(1-\pi)}{(\pi(R - r_\infty) - \delta)\beta\gamma}. \tag{C.4}$$

The equation (D.2) does not change. The new value of R is given by:

$$R = \frac{\delta(1-\beta) + \delta\gamma r_\infty(1-\pi)}{\pi(1-\beta)}. \tag{C.5}$$

The link between A and b also changes by a factor γ:

$$b = A\frac{-\delta - \delta v\gamma + \pi R + \pi(R - r_\infty)\gamma v}{\gamma(\delta - \pi(R - r_\infty))} \tag{C.6}$$

We take the same parameters values as before and draw the values of our valuation components. R behaves the same way as the one with the MW constraint. There are some problems for v because v remains negative for any value of R. This means that $v < \frac{1}{\beta}$ and the KM constraint rules out the non-linear pricing, as in the previous chapter.

Appendix D

Affine pricing with Miao and Wang constraint

As proved in chapter 4, following Miao and Wang (2011), there exists steady state of the model in which $b \neq 0$. Any firm has a positive b shift in the price, which creates a distortion, especially for firms with low capital. This originality of pricing comes from the use of the MW constraint. We check in our long-term debts model whether the use of this constraint may create such a non-linear pricing.

We compute the steady state values when the bubble component is not zero. The equation (5.46) gives the value of v:

$$v = \frac{1 - \beta - (R - r_\infty)(1 - \pi)}{\beta((R - r_\infty)\pi - \delta)}. \tag{D.1}$$

From its own equation, we also have a value of v:

$$\beta\gamma v^2(\delta - \pi(R - r_\infty)) + v - v\gamma(R - r_\infty)(1 - \pi) - v\beta\pi R - \beta v(1 - \delta) - R(1 - \pi) = 0. \tag{D.2}$$

These two expressions of v lead to the value of R:

$$R = \frac{(1 - \gamma(1 - \beta) - \beta(1 - \delta))\,(1 - \beta + r_\infty(1 - \pi))}{\beta\pi(1 - \beta) + (1 - \pi)\,(1 - \beta\delta - \gamma(1 - \beta) - \beta(1 - \delta))}. \tag{D.3}$$

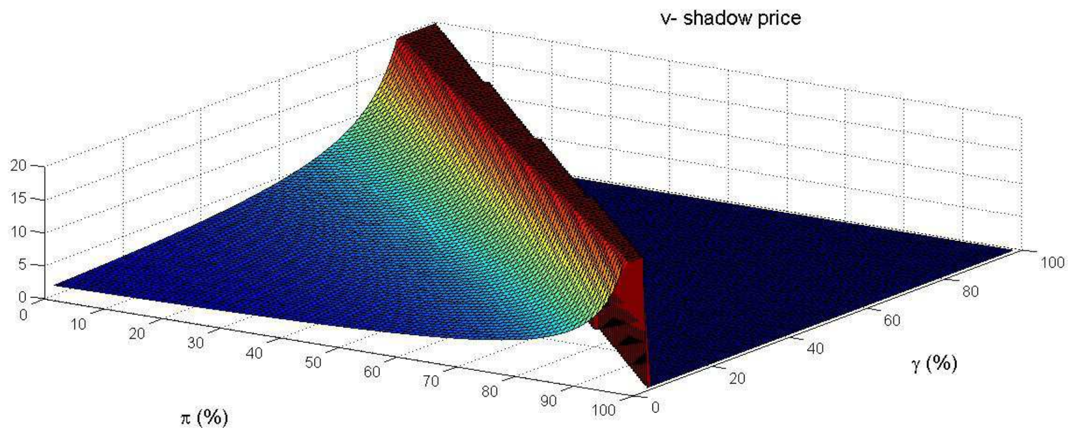

Figure D.1: Shadow price v of the non-linear valuation with respect to π and γ

The value of v is given by the value of R and the link between b and A comes from equation (5.48):

$$b = A\frac{-\delta - \delta v\gamma + \pi R + \pi(R - r_\infty)\gamma v}{\delta - \pi(R - r_\infty)}. \tag{D.4}$$

The equation (5.50) together with (D.4) determines A and b. We represent the main values of b and v. We take $\alpha = 0.4$, $\delta = 0.025$, $\beta = 0.96$ and $r = 4\%$. On the values of v, we limit to an arbitrary value (20) because it also diverges like in the linear pricing. The dark area is forbidden because $v \leq \frac{1}{\beta}$.

The shadow price of the non-linear pricing does not exist for the same values as the shadow price of the linear pricing: low values of the investment probabilities are privileged. This corresponds to the results of the model of Miao and Wang (2011).

We represent the bubble component on Figure D.2, again the dark value corresponds to the negative values of b. Surprisingly the bubble is increasing with respect to π which contrasts with the previous chapter. This may correspond to acceleration phenomena, such as new technologies, for example the dot-com bubble.

We must also check that the value of the rental rate of the capital verifies $R > r_\infty = 4\%$: this condition is satisfied while $\gamma > 25\%$ or $\pi < 75\%$, on Figure D.3.

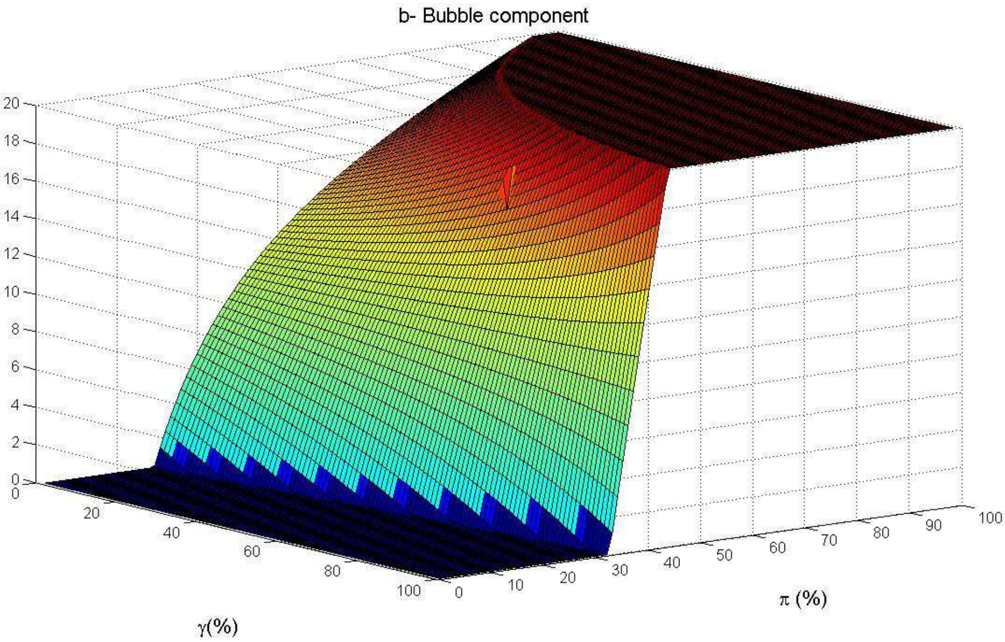

Figure D.2: Affine component of the valuation b, non-linear pricing, with respect to π and γ

The last point is the average value of the equity E that needs to remain above 0 on Figure D.4:

We intersect all the previous domains to get the complete zone on Figure D.5 where the non-linear pricing exists in the economy.

Everywhere this steady state exists, it is unstable: the system of the 3 linearized equations has 3 eigenvalues larger than 1 in modulus. The prices can never reach this equilibrium. For all values of the parameters π and γ such that the non-linear pricing exists, the equation (5.46) on the shift term b does not verify the transversality condition if $b \neq 0$:

$$(R_t - r_t)(1 - \pi + \pi\beta v_{t+1}) - \delta\beta v_{t+1} > 0. \tag{D.5}$$

This shows that there is no equilibirum affine pricing using Miao and Wang constraint.

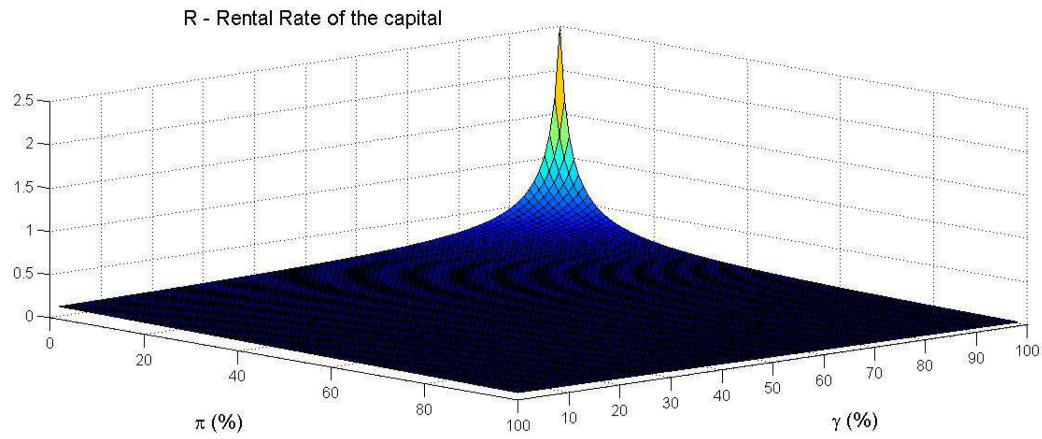

Figure D.3: Rental rate of the capital R, non-linear pricing, with respect to π and γ

When the capital of firms is composed of equity and debt, the choice of the borrowing constraint does not matter, the prices of firms are the same, and there exists only a unique linear pricing when stochastic investment opportunities force the firms to invest "as much as possible".

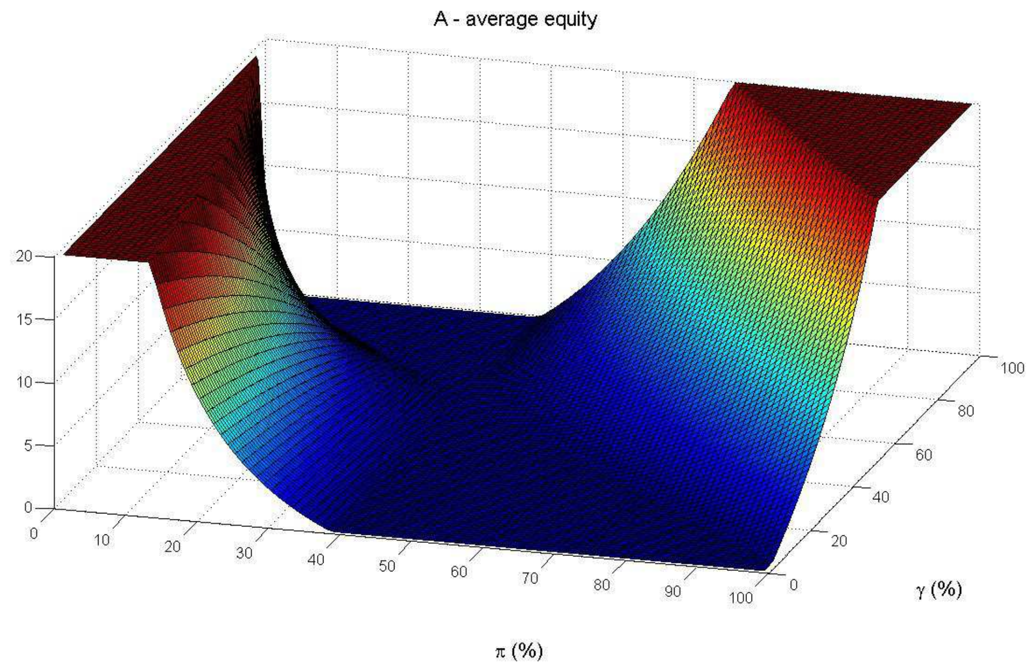

Figure D.4: Average equity E, non-linear pricing, with respect to π and γ

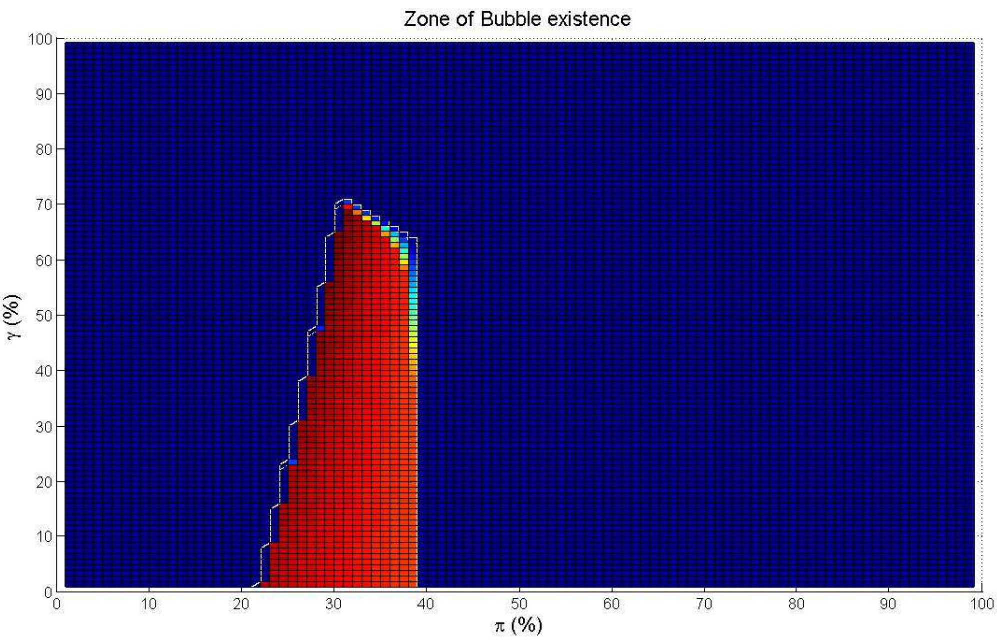

Figure D.5: Existence domain of the non-linear pricing, with respect to π and γ

General Conclusion

This dissertation proposes a multidisciplinary approach to the recent crises. The analysis deals with two key issues of crises: the connectivity of financial activities and the bubbles on assets before the crises. To study these aspects, we have been using varied methods, among which, network and graph theories, Bellman equations, Matlab and Dynare computing. The research is also "restricted to" a standard context of rational agents within equilibrium markets, especially when there are some kind of bubbles.

The global view of the financial network is presented in Chapter 2. By using a new representation of the financial network, we have been able to model the bilateral interactions between the financial agents and the network. We have built our network on real connectivity, trying to capture the intermediate connected state of financial activities: complete or sparse networks are not realistic. We have deduced our results by simulating the network, which in return allowed to let connectivity vary over a large range. Depending on the time-horizon maximization of the agents, we have proved that financial networks may reach a more or less connected equilibrium. When there are a lot of financial connections, there exists systemic risk, but the systemic crisis probability remains low. When agents decide to make only a few connections, because they fear defaults, there is no systemic risk, but little profits, and more strategically defaulting agents. Intermediate connected states are avoided because they tend to propagate defaults easily, and therefore do not guarantee interesting payoffs. We have highlighted the problems linked to myopic agents: when they are enough capitalized, they build a highly connected network, which is subject to a significant systemic risk. The regulator may intervene to avoid such situations by properly settling a prudential ratio. However, this ratio must be wisely employed, because it may generate liquidity problems.

Our network approach could be somehow improved by allowing agents of different sizes. This could be done by introducing a random variable on the capitalization of the agents, or directly by choosing more capitalized agents. In this case, the model would account for too big too fail institutions. However, the closer we get to real networks, the more difficult it becomes to derive the properties (behavior of agents) of the network. Another extension of the model would be to consider risk-averse agents. Because staying in the network includes a part of risk (becoming contaminated by defaulting agents, face multiple defaulting counterparties), a concave utility function would lead to more liquidity problems and a higher systemic risk, even when agents optimize in the long-run. Nevertheless, such a modification would obsucre understanding of the model and would not deliver new policy implications.

To deal with bubbles, our approach brought out the significance of collateral questions for borrowing and the necessity to introduce a stochastic investment to generate linear prices of firms: recent research have demonstrated that when the price of production firms guarantee loans, asset prices are likely to include a bubble component. This bubble component in the prices permits the agents to borrow more by relaxing the borrowing constraint. Our contribution was to apprehend the effect of interest rates on such bubbles. In chapter 4, we started from a recent article (Miao and Wang, 2011), where firms face stochastic investment opportunities and borrowing constraints. Firms borrow when investment is possible, to reach the optimal investment level. As a result, prices of firms may include a bubble component. We adapted this model with a net positive interest rate on the short-term debts. We proved that the effect of interest rates on prices and capital was not significant. Furthermore, equilibrium prices were affine with respect to the capitalization of firms and did not really include a bubble component. We decided to change the capital structure of firms, by introducing a permanent part of long-term debt in the capital of firms, in chapter 5. This new idea was successful: we proved that two equilibrium prices coexist for firms. One of them corresponds to a higher average level of capital, where firms' prices correspond to the Tobins' Q. In this case, the level of equity is high enough, such that firms are not penalized by the stochastic investment. The other equilibrium corresponds to a lower average capital, and maximal investment of firms. In this equilibrium, firms' prices are higher, and help firms to reach the optimal amount of investment. Furthermore, these prices are very sensitive to the interest rates on loans. Positive interest-rate shocks decrease the equity, but increase the price, allowing to keeps a high level of debt. When interest rate shocks are persistent, firms' prices may be discontinuous, and somehow illustrate market's crashes. Even if price of firms exceed their usual Tobin's Q value, there are no bubbles in prices as soon as the equity is positive. Nevertheless, an unusual situation may happen depending on the investments parameters: the equity does not exist, but has a net positive price, this is a bubble. The bubble allows for borrowing, and with the debt used as capital, firms can generate strictly positive cash flows. In this model of long-term debts in the capital of firms, we also proved that the choice of the borrowing constraint does not influence the equilibrium prices. In addition, affine prices of firms do not exist either.

There are many ways to deepen the analysis of this model. First, it would be relevant to get closer to the bubble, even if asymptotic properties (zero equity, infinite shadow price, continuous price) represent a technical challenge. Then, it would be interesting to investigate a global equilibrium in which the financial intermediaries collect the households deposits' and lend them to the firms. This would endogenize the supply of loans, and the interest rate. By limiting the supply of loans, pricings of firms could be different. Finally, to submit this long-term debts model to a review, we could recreate a two-state problem, by introducing a probability of falling from the linear-price to the Q-price.

Because this research concludes that bubbles are somewhat rare to appear in equilibrium markets with rational agents, the research framework could be re-examined. It was necessary to start with this particular framework to understand how we can broaden it step-by-step to the diversified set of other models. For example, instead of introducing *ad hoc* irrational agents, it is more desirable to keep rational agents, but to allow them some heterogeneity in their sets of information, the timing of information, beliefs... Also models with multiple equilibria or non-equilibrium markets are well-suited to recreate bubbles and crashes. Even if they outperform classic models, how can we understand and test the validity of the hypotheses, how can we improve their tractability, and how to standardize them to use them in financial institutions? In the extreme, behavioral finance is very specialized, and almost irreconcilable with any other approach.

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Essays on financial crisis

Gwenaël Moysan

Résumé

La thèse traite de differents aspects des crises financières et de leur gestion par les régulateurs.

Les systèmes financiers complexes, tel le réseau interbancaire, peuvent être modélisés par une approche de type réseau, pour calculer la propagation des faillites et modéliser le risque systémique. Partant d'un réseau d'agents identiquement capitalisés, l'impact d'un ratio de capitalisation est testé selon l'horizon de maximisation des profits des agents. Performante lorsque les agents optimisent en horizon infini, la prévention du risque systémique par ce ratio peut créer d'importants problèmes de liquidité lorsque les agents sont myopes.

L'effet des taux d'intérêts sur les bulles de crédit est analysé en partant d'un modèle basé sur une économie de production, dans laquelle les entreprises font face à de rares opportunités d'investissement. Lorsqu'elles investissent, ces entreprises ont recours à de la dette court-terme, limitée à une partie de leur valeur boursière. Dans ce modèle générant des bulles sur les prix des firmes, les taux d'intérêts sur la dette n'ont que peu d'effets sur les prix des firmes. En outre, des réserves sont émises concernant la présence de véritables bulles dans les prix vis-à-vis des concepts historiques.

Le précédent modèle est étendu en introduisant une part de dette à long-terme dans le capital des firmes. Dans ce cadre, les valeurs boursières des firmes sont très réactives au taux d'intérêt, étant même discontinues lors de chocs de taux persistants. Par ailleurs, l'économie de production peut atteindre un état bullier : les prix des firmes reflètent les gains de capital uniquement dus à la dette.

Mots Clés : Réseau financier, contagion, risque systémique, bulles, taux d'intérêt, valorisation, capital, collatéral, investissement stochastique, ratios prudentiels.

Complex financial systems, such as the interbank network, can be naturally captured using a network approach. This allows to calculate contamination of defaults and to model systemic risk. Our network is composed of identically capitalized agents. The effect of a capitalization ratio is determined depending on the maximization horizon of the agents : short-term, myopic or long-term. When agents optimize their payoffs in the long-run, the capitalization ratio is fully effective and prevents systemic risk. However, when agents adopt a myopic behavior, the capitalization ratio may trade systemic risk for liquidity scarcity.

Starting from a production economy, in which firms face stochastic investment opportunities, we study the impact of the interest rates on bubbles in firms' prices. Capital of firms is exclusively made of equity, but when facing an investment opportunity, firms may borrow. Precisely, firms access short-term debts, and the amount of the debt is limited to a fraction of the price of their equities. This model seems to recreate bubbles on firms' prices. Unfortunately, interest rates do not affect prices to a large extent, and we may question whether prices of firms include a bubble component, with respect to the standard definition of bubbles : the discounted sum of the incoming cash flows.

This previous model is extended by allowing firms to have a permanent debt. Actually, capital of firms is composed of equity and debt. In this case, firms' prices are very sensitive to interest rates, and may be discontinuous when interest rate shocks last over the periods. This model also exhibits a purely bubbly state : prices of firms only represent capitals profits generated by debts, there is no equity.

Keywords : financial network, contagion, systemic risk, bubbles, interest rates, pricing, equity, collateral, stochastic investment, prudential ratios.